

Transmission eigenvalue-free regions

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Abstract. We prove the existence of large regions free of eigenvalues of the interior transmission problem.

1 Introduction and statement of results

Let $\Omega \subset \mathbf{R}^d$, $d \geq 2$, be a bounded, connected domain with a C^∞ smooth boundary $\Gamma = \partial\Omega$. A complex number $\lambda \in \mathbf{C}$, $\lambda \neq 0$, will be said to be a transmission eigenvalue if the following problem has a non-trivial solution:

$$\begin{cases} (\nabla c_1(x) \nabla + \lambda n_1(x)) u_1 = 0 & \text{in } \Omega, \\ (\nabla c_2(x) \nabla + \lambda n_2(x)) u_2 = 0 & \text{in } \Omega, \\ u_1 = u_2, \quad c_1 \partial_\nu u_1 = c_2 \partial_\nu u_2 & \text{on } \Gamma, \end{cases} \quad (1.1)$$

where ν denotes the exterior Euclidean unit normal to Γ , $c_j, n_j \in C^\infty(\overline{\Omega})$, $j = 1, 2$ are strictly positive real-valued functions. The purpose of this work is to study the localization of the possible transmission eigenvalues on \mathbf{C} as $|\lambda| \rightarrow \infty$ under the condition

$$c_1(x)n_1(x) \neq c_2(x)n_2(x), \quad \forall x \in \Gamma. \quad (1.2)$$

Our first result is the following

Theorem 1.1 *Assume (1.2) together with the condition*

$$c_1(x) = c_2(x), \quad \partial_\nu c_1(x) = \partial_\nu c_2(x), \quad \forall x \in \Gamma. \quad (1.3)$$

Then there are no transmission eigenvalues in $\Lambda^- \cup \Lambda^+$, where

$$\Lambda^+ := \left\{ \lambda \in \mathbf{C} : \operatorname{Re} \lambda \geq 0, |\operatorname{Im} \lambda| \geq C_\varepsilon (\operatorname{Re} \lambda + 1)^{\frac{3}{4} + \varepsilon} \right\}$$

for every $0 < \varepsilon \ll 1$ with $C_\varepsilon > 0$,

$$\Lambda^- := \left\{ \lambda \in \mathbf{C} : \operatorname{Re} \lambda \leq -\tilde{C} \right\} \cup \left\{ \lambda \in \mathbf{C} : -\tilde{C} \leq \operatorname{Re} \lambda \leq 0, |\operatorname{Im} \lambda| \geq C \right\}$$

$C, \tilde{C} > 0$ being constants.

Remark 1. In the case $c_1 \equiv c_2 \equiv 1$ it was proved previously in [7] that outside any sector $|\operatorname{Im} \lambda| \leq \theta \operatorname{Re} \lambda$, $\forall \theta > 0$, there are at most a finite number of transmission eigenvalues. In the

case $c_1 \equiv c_2 \equiv 1$, $n_1 \equiv 1$ and $n_2(x) > 1$, $\forall x \in \overline{\Omega}$, the above theorem is proved in [3] but with Λ^+ replaced by a smaller eigenvalue-free region of the form

$$\left\{ \lambda \in \mathbf{C} : \operatorname{Re} \lambda \geq 0, |\operatorname{Im} \lambda| \geq C (\operatorname{Re} \lambda + 1)^{\frac{24}{25}} \right\}.$$

The situation is far more interesting and different when the condition (1.3) is not fulfilled. In this case we have the following

Theorem 1.2 *Assume (1.2) together with the condition*

$$c_1(x) \neq c_2(x), \quad \forall x \in \Gamma. \quad (1.4)$$

Then, there are no transmission eigenvalues in

$$\left\{ \lambda \in \mathbf{C} : \operatorname{Re} \lambda \geq 0, |\operatorname{Im} \lambda| \geq C (\operatorname{Re} \lambda + 1)^{\frac{4}{5}} \right\}$$

with a constant $C > 0$. Moreover, if in addition we assume either the condition

$$\frac{n_1(x)}{c_1(x)} \neq \frac{n_2(x)}{c_2(x)}, \quad \forall x \in \Gamma, \quad (1.5)$$

or the condition

$$\frac{n_1(x)}{c_1(x)} = \frac{n_2(x)}{c_2(x)}, \quad \forall x \in \Gamma, \quad (1.6)$$

then there are no transmission eigenvalues in Λ^+ . Under the condition

$$(c_1(x) - c_2(x))(c_1(x)n_1(x) - c_2(x)n_2(x)) > 0, \quad \forall x \in \Gamma, \quad (1.7)$$

there are no transmission eigenvalues in Λ^- . Finally, if we assume the condition

$$(c_1(x) - c_2(x))(c_1(x)n_1(x) - c_2(x)n_2(x)) < 0, \quad \forall x \in \Gamma, \quad (1.8)$$

then for every $N \geq 1$ there is a constant $C_N > 0$ so that there are no transmission eigenvalues in

$$\left\{ \lambda \in \mathbf{C} : \operatorname{Re} \lambda \leq 0, |\operatorname{Im} \lambda| \geq C_N (|\operatorname{Re} \lambda| + 1)^{-N} \right\}.$$

Remark 2. Note that the condition (1.8) implies (1.5).

Remark 3. It is clear from the proof that the fact that we can take an arbitray N above comes from the C^∞ - smoothness of the boundary Γ and the coefficients c_j , n_j near Γ . Therefore, it is natural to expect that if more regularity is assumed (e.g. Gevrey class or analyticity), a larger eigenvalue-free region exists. Indeed, using the techniques of [13] one can show that in the analytic case there is a region free of eigenvalues of the form

$$\left\{ \lambda \in \mathbf{C} : \operatorname{Re} \lambda \leq 0, |\operatorname{Im} \lambda| \geq C \exp \left(-\beta |\operatorname{Re} \lambda|^{1/2} \right) \right\}$$

with some constants $C, \beta > 0$.

Remark 4. It is clear from our construction of the parametrix in the region $\operatorname{Re} \lambda < 0$ that under the condition (1.8) one can construct quasimodes for the problem (1.1) concentrated in an arbitrary neighbourhood of the boundary Γ due to the existence in this case of surface

waves moving on Γ with a speed $\sqrt{c(x)}$, where c denotes the restriction on Γ of the function $|c_1^2 - c_2^2|/|c_1 n_1 - c_2 n_2|$. These waves are very similar to the Rayleigh surface waves in the linear elasticity studied in [14], [15], [16] and have practically the same properties. In particular, as in [15] one can show that these quasimodes imply the existence of infinitely many transmission eigenvalues with negative real parts. In fact, much more can be proved, namely an asymptotic of the counting function $N^-(r) = \#\{\lambda - \text{trans. eig.} : \text{Re } \lambda < 0, |\lambda| \leq r^2\}$, $r > 1$. Indeed, as in [14] one can show that

$$N^-(r) = \left(\frac{r}{2\pi}\right)^{d-1} \omega_{d-1} \int_{\Gamma} c(x)^{-\frac{d-1}{2}} dx + O(r^{d-2})$$

where $\omega_{d-1} := \text{Vol}\{x \in \mathbf{R}^{d-1} : |x| \leq 1\}$, provided the multiplicity of an eigenvalue λ_k is defined by

$$\text{mult}(\lambda_k) = \text{tr} (2i\pi)^{-1} \int_{|\lambda - \lambda_k| = \varepsilon} \left(c_1 \frac{d\mathcal{N}_1}{d\lambda}(\lambda) - c_2 \frac{d\mathcal{N}_2}{d\lambda}(\lambda) \right) (c_1 \mathcal{N}_1(\lambda) - c_2 \mathcal{N}_2(\lambda))^{-1} d\lambda,$$

$0 < \varepsilon \ll 1$, where $\mathcal{N}_j(\lambda)$ denotes the Dirichlet-to-Neumann map corresponding to the pair (n_j, c_j) .

Remark 5. These kind of eigenvalue-free regions are crucial for bounding the remainder in the asymptotics of the counting function of all transmission eigenvalues. Indeed, it has been proved recently in [10] that the total counting function $N(r) = \#\{\lambda - \text{trans. eig.} : |\lambda| \leq r^2\}$, $r > 1$, satisfies the asymptotics

$$N(r) = (\tau_1 + \tau_2)r^d + O_\varepsilon(r^{d-\kappa+\varepsilon}), \quad \forall 0 < \varepsilon \ll 1,$$

where $0 < \kappa \leq 1$ is such that there are no transmission eigenvalues in the region

$$\left\{ \lambda \in \mathbf{C} : |\text{Im } \lambda| \geq C (|\text{Re } \lambda| + 1)^{1-\frac{\kappa}{2}} \right\}, \quad C > 0,$$

and

$$\tau_j = \frac{\omega_d}{(2\pi)^d} \int_{\Omega} \left(\frac{n_j(x)}{c_j(x)} \right)^{d/2} dx,$$

ω_d being the volume of the unit ball in \mathbf{R}^d .

Corollary 1.3 *Assume (1.2) together with the condition*

$$n_1(x) = n_2(x), \quad \forall x \in \Gamma. \tag{1.9}$$

Then there are no transmission eigenvalues in $\Lambda^- \cup \Lambda^+$.

There have been recently many works studying mainly the discreteness and the asymptotic behaviour of the counting function of the transmission eigenvalues (see [2], [3], [5], [6], [7], [8], [9], [11], [12], [17] and the references therein). For example, in [5] Weyl type asymptotics have been proved for the counting function of all transmission eigenvalues under the condition (1.4), while in [8] a lower bound of right order has been proved for the counting function of the transmission eigenvalues belonging to $(0, +\infty)$. Under the conditions (1.2) and (1.3), upper bounds of the counting function of all transmission eigenvalues have been proved in [2], [11], [12], and an asymptotic has been given in [9] when Ω is a ball and the coefficients are constants.

To prove the eigenvalue-free regions we transform the problem (1.1) into a semi-classical one by putting $h = |\operatorname{Re} \lambda|^{-1/2}$, $z = \frac{\lambda}{|\operatorname{Re} \lambda|}$, if $|\operatorname{Re} \lambda| \geq |\operatorname{Im} \lambda|$, $|\operatorname{Re} \lambda| \gg 1$, and $h = |\operatorname{Im} \lambda|^{-1/2}$, $z = \frac{\lambda}{|\operatorname{Im} \lambda|}$, if $|\operatorname{Im} \lambda| \geq |\operatorname{Re} \lambda|$, $|\operatorname{Im} \lambda| \gg 1$. Thus we have to show that if h is small enough and z/h^2 belongs to the eigenvalue-free regions described above, then under the corresponding conditions the solutions to (1.1) are identically zero. In fact, it suffices to show that $u_1|_\Gamma$ is identically zero since this would imply that u_1 and u_2 are identically zero, too. To do so, we construct in Section 3 a parametrix of the solutions to the interior boundary value problem (see equation (3.1) below) near the boundary Γ by using h -FIOs with a complex-valued phase satisfying the eikonal equation mod $O(x_1^N)$, $N > 1$ being an arbitrary integer and $0 < x_1 \ll 1$ is the normal coordinate to the boundary (which is nothing else but the Euclidean distance from a point $x \in \Omega$ to Γ). The amplitude must satisfy the transport equations mod $O(x_1^N)$. We solve these equations in Section 4. Furthermore, we use this parametrix to show that the Dirichlet-to-Neumann map can be approximated by h - Ψ DOs belonging (uniformly in z) to the class \mathcal{S}_0^1 (see Section 2 for the definition) if $\operatorname{Re} z = -1$, $|\operatorname{Im} z| \leq 1$ or $|\operatorname{Re} z| \leq 1$, $|\operatorname{Im} z| = 1$, and to $\mathcal{S}_{1/2-\epsilon}^1$ if $\operatorname{Re} z = 1$, $h^{\frac{1}{2}-\epsilon} \leq |\operatorname{Im} z| \leq 1$, $0 < \epsilon \ll 1$. Thus we reduce the problem of finding eigenvalue-free regions to that one of inverting h - Ψ DOs (depending on an additional parameter z) on a compact manifold. Note that these classes of h - Ψ DOs are nice in the sense that there is a symbol calculus for them as well as a simple criteria of L^2 boundeness (e.g. see [1]). We recall these properties of the h - Ψ DOs in Section 2. In particular, to invert such an operator it suffices to invert its principal symbol and determine the class of symbols the inverse belongs to. That is precisely what we do in Section 5. Note that the study of the case $n = 1$ in [17] (see also [9]) suggests that there are probably larger eigenvalue-free regions in $\operatorname{Re} \lambda > 0$ at least in some specific cases as for example Ω is a ball and c_j, n_j constants. In this latter case one has to invert Bessel functions instead of h - Ψ DOs, which seems to be much easier. In the general case studied here, however, it would be impossible to do better since it is impossible to construct a parametrix for the equation (3.1) when $\operatorname{Re} z = 1$, $0 < |\operatorname{Im} z| \ll h^{\frac{1}{2}-\epsilon}$. As a consequence, in this region the Dirichlet-to-Neumann map is no longer an h - Ψ DO, and hence it is impossible to use the theory of the h - Ψ DOs to invert our operator.

2 h -pseudo-differential operators on a compact manifold

Let X be a C^∞ smooth compact manifold without boundary, $n = \dim X \geq 1$. Let (x, ξ) be coordinates on T^*X and let $a \in C^\infty(T^*X)$. Then the h -pseudo-differential operator with a symbol a is defined as follows

$$(\operatorname{Op}_h(a)f)(x) = \left(\frac{1}{2\pi h}\right)^n \int_{T^*X} e^{-\frac{i}{h}\langle x-y, \xi \rangle} a(x, \xi) f(y) dy d\xi$$

where $h > 0$ is a small parameter. Of course, in order that this operator has *nice* properties the function a must belong to some class of symbols. In what follows in this section we will introduce several classes of symbols which will play important role in our analysis. First, given $\ell \in \mathbf{R}$, $\delta_1, \delta_2 \geq 0$ and a function $\mu > 0$, we denote by $S_{\delta_1, \delta_2}^\ell(\mu)$ the set of all functions $a \in C^\infty(T^*X)$ such that

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leq C_{\alpha, \beta} \mu^{\ell - \delta_1|\alpha| - \delta_2|\beta|}$$

for all multi-indices α, β with constants $C_{\alpha, \beta} > 0$ independent of h, μ . The following simple properties will be often used in the next sections: If $a \in S_{\delta_1, \delta_2}^\ell(\mu)$, then $\partial_x^\alpha \partial_\xi^\beta a \in S_{\delta_1, \delta_2}^{\ell - \delta_1|\alpha| - \delta_2|\beta|}(\mu)$.

If $a_j \in S_{\delta_1, \delta_2}^{\ell_j}(\mu)$, $j = 1, 2$, then $a_1 a_2 \in S_{\delta_1, \delta_2}^{\ell_1 + \ell_2}(\mu)$. If $b(x) \in C^\infty(X)$, independent of ξ , and $a \in S_{\delta_1, \delta_2}^\ell(\mu)$, then $ba \in S_{\delta_1, \delta_2}^\ell(\mu)$ if $\mu \leq \text{Const}$ or $\mu \geq \text{Const} > 0$ and $\delta_1 = 0$. We also need a simple criteria for this class of operators to be bounded on $L^2(X)$.

Proposition 2.1 *Let the function a satisfy*

$$\sup_{x, \xi \in T^*X} |\partial_x^\alpha a(x, \xi)| = C_\alpha < \infty \quad (2.1)$$

for all multi-indices α . Then the operator $\text{Op}_h(a)$ is bounded on $L^2(X)$ and

$$\|\text{Op}_h(a)\|_{L^2(X) \rightarrow L^2(X)} \leq C \sum_{|\alpha| \leq n+1} C_\alpha h^{|\alpha|/2} \quad (2.2)$$

with a constant $C > 0$ independent of h and C_α . In particular, if $a \in S_{\delta, \delta_2}^\ell(\mu)$ with $\ell \leq 0$ and $\mu(x, \xi) \geq \mu_0 > 0$, we have the bound

$$\|\text{Op}_h(a)\|_{L^2(X) \rightarrow L^2(X)} \leq C \mu_0^\ell \left(1 + \frac{\sqrt{h}}{\mu_0^\delta}\right)^{n+1} \quad (2.3)$$

with a constant $C > 0$ independent of h and μ_0 .

Proof. It is based on the observation that the boundness of $\text{Op}_h(a)$ on L^2 is equivalent to that of the classical operator $\text{Op}_1(a_h)$, where $a_h(x, \xi) = a(\sqrt{h}x, \sqrt{h}\xi)$. On the other hand, since X is compact, it is well known (see Theorem 18.1.11' of [4]) that the norm of $\text{Op}_1(a_h) : L^2 \rightarrow L^2$ is bounded by $\sum_{|\alpha| \leq n+1} \sup |\partial_x^\alpha a_h(x, \xi)|$, which implies (2.2). \square

Given $k \in \mathbf{R}$, $0 \leq \delta \leq \frac{1}{2}$, denote by \mathcal{S}_δ^k the set of all functions $a \in C^\infty(T^*X)$ satisfying

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leq C_{\alpha, \beta} h^{-\delta(|\alpha| + |\beta|)} \langle \xi \rangle^{k - |\beta|}$$

for all multi-indices α, β with constants $C_{\alpha, \beta} > 0$ independent of h . We will denote by OPS_δ^k the set of the h -pseudo-differential operators with symbols in \mathcal{S}_δ^k . It follows from the above proposition that if $a \in \mathcal{S}_\delta^0$, then $\text{Op}_h(a) : L^2 \rightarrow L^2 = O(1)$. It is also well-known (e.g. see Section 7 of [1]) that when $\delta < \frac{1}{2}$ there is a nice symbol calculus and in particular the symbol of the composition of h -pseudo-differential operators with symbols in this class can be calculated explicitly mod $O(h^\infty)$. Thus, if $a \in \mathcal{S}_\delta^k$ with $0 \leq \delta < \frac{1}{2}$ and $|a| \geq C \langle \xi \rangle^k$ with $C > 0$ independent of h , then the operator $\text{Op}_h(a)$ is invertible with an inverse belonging to OPS_δ^{-k} . The following proposition is essentially proved in Section 7 of [1]. Here we sketch the proof for the sake of completeness.

Proposition 2.2 *Let $h^{\ell_\pm} a^\pm \in \mathcal{S}_\delta^{\pm k}$, $\delta < \frac{1}{2}$, where $\ell_\pm \geq 0$ are some numbers. Assume in addition that the functions a^\pm satisfy*

$$\left| \partial_x^{\alpha_1} \partial_\xi^{\beta_1} a^+(x, \xi) \partial_x^{\alpha_2} \partial_\xi^{\beta_2} a^-(x, \xi) \right| \leq \mu_0 C_{\alpha_1, \beta_1, \alpha_2, \beta_2} h^{-\frac{|\alpha_1| + |\beta_1| + |\alpha_2| + |\beta_2|}{2}} \quad (2.4)$$

for all multi-indices $\alpha_1, \beta_1, \alpha_2, \beta_2$ such that $|\alpha_j| + |\beta_j| \geq 1$, $j = 1, 2$, with constants $C_{\alpha_1, \beta_1, \alpha_2, \beta_2} > 0$ independent of h , μ_0 . Then we have

$$\|\text{Op}_h(a^+) \text{Op}_h(a^-) - \text{Op}_h(a^+ a^-)\|_{L^2(X) \rightarrow L^2(X)} \leq C \mu_0 + Ch \quad (2.5)$$

with a constant $C > 0$ independent of h and μ_0 .

Proof. In view of formula (7.15) of [1] the operator in the left-hand side of (2.5) whose norm we would like to bound is an h -psdo with symbol $b(x, \xi, x, \xi)$, where the function b is given by

$$b(x, \xi, y, \eta) = \left(e^{ihD_\xi \cdot D_y} - 1 \right) a(x, \xi, y, \eta)$$

where we have put $a = a^+(x, \xi)a^-(y, \eta)$ and $D = -i\partial$. It follows from the analysis in Section 7 of [1] (see (7.17) and (7.19)) that given any integer $N \geq 2$ the function b can be decomposed as $b_N + \tilde{b}_N$, where

$$b_N = \sum_{j=1}^{N-1} \frac{1}{j!} (ihD_\xi \cdot D_y)^j a = \sum_{j=1}^{N-1} \frac{(ih)^j}{j!} \sum_{|\alpha|=j} D_\xi^\alpha a^+(x, \xi) D_y^\alpha a^-(y, \eta)$$

while the remainder \tilde{b}_N satisfies

$$\left| \partial_x^\alpha \partial_y^\beta \tilde{b}_N(x, \xi, y, \eta) \right| \leq C_{\alpha, \beta} h^{N(1-2\delta)-\ell} \langle \xi \rangle^k \langle \eta \rangle^{-k} \leq C_{\alpha, \beta} h^{N(1-2\delta)-\ell}$$

if $\eta = \xi$, where $\ell = \ell_+ + \ell_- + s_n + \delta(|\alpha| + |\beta|)$ is independent of N . In view of Proposition 2.1, this implies that there exists some $\ell_1 > 0$ independent of N such that

$$\left\| \text{Op}_h(\tilde{b}_N(x, \xi, x, \xi)) \right\|_{L^2 \rightarrow L^2} \leq Ch^{N(1-2\delta)-\ell_1} \leq Ch \quad (2.6)$$

if N is taken large enough. On the other hand, it is easy to see that (2.4) implies

$$\left| \partial_x^\alpha \partial_y^\beta b_N(x, \xi, x, \xi) \right| \leq \mu_0 C_{\alpha, \beta} h^{-\frac{|\alpha|+|\beta|}{2}}.$$

By Proposition 2.1,

$$\left\| \text{Op}_h(b_N(x, \xi, x, \xi)) \right\|_{L^2 \rightarrow L^2} \leq C\mu_0. \quad (2.7)$$

Clearly, (2.5) follows from (2.6) and (2.7). \square

3 Parametrix near the boundary

Let $z \in Z = Z_1 \cup Z_2 \cup Z_3$, where $Z_1 = \{z \in \mathbf{C} : \text{Re } z = 1, 0 < |\text{Im } z| \leq 1\}$, $Z_2 = \{z \in \mathbf{C} : \text{Re } z = -1, |\text{Im } z| \leq 1\}$, $Z_3 = \{z \in \mathbf{C} : |\text{Re } z| \leq 1, |\text{Im } z| = 1\}$. Clearly, we have $1 \leq |z| \leq 2$. Given any $f \in L^2(\Gamma)$ let u solve the equation

$$\begin{cases} (P(h) - z)u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Gamma, \end{cases} \quad (3.1)$$

where

$$P(h) = -\frac{h^2}{n(x)} \nabla c(x) \nabla$$

and $h > 0$ is a small parameter, $c, n \in C^\infty(\overline{\Omega})$ being strictly positive functions. Let (x', ξ') be coordinates on $T^*\Gamma$ and denote by $r_0(x', \xi')$ the principal symbol of the Laplace-Beltrami operator, $-\Delta_\Gamma$, on Γ equipped with the Riemannian metric induced by the Euclidean one in \mathbf{R}^d . It is well-known that r_0 is a polynomial function in ξ' , homogeneous of order 2, and $C_2|\xi'|^2 \geq r_0(x', \xi') \geq C_1|\xi'|^2$ with constants $C_2 > C_1 > 0$. Set $m(x) = \frac{n(x)}{c(x)}$ and denote by γ the

restriction on Γ , that is, $\gamma m = m|_{\Gamma}$. Define the function $\rho \in C^\infty(T^*\Gamma)$ as being the root of the equation

$$\rho^2 + r_0(x', \xi') - \gamma m(x')z = 0$$

with $\text{Im } \rho > 0$ (which is easily seen to exist as long as $z \in Z$). In what follows in this paper C and \tilde{C} will denote positive constants independent of z , h and f , which may change from line to line.

Lemma 3.1 *Let $z \in Z_1 \cup Z_3$. Then*

$$\text{Im } \rho \geq \frac{|\text{Im } z|}{2|\rho|}, \quad (3.2)$$

$$|\rho| \geq C\sqrt{|\text{Im } z|}, \quad (3.3)$$

while for $r_0 \geq 2\gamma m$, we have

$$\tilde{C}\sqrt{r_0 + 1} \geq 2\text{Im } \rho \geq |\rho| \geq C\sqrt{r_0 + 1}. \quad (3.4)$$

Let $z \in Z_2$. Then (3.4) holds for all $r_0 \geq 0$.

Proof. Clearly, (3.2) follows from the identity

$$2\text{Im } \rho \text{Re } \rho = \gamma m \text{Im } z.$$

The bound (3.3) follows easily from the identity

$$|\rho|^4 = |r_0 - \gamma m z|^2 = (r_0 - \gamma m \text{Re } z)^2 + (\gamma m \text{Im } z)^2.$$

For $r_0 \geq 2\gamma m$, we have

$$|\rho|^4 \geq \frac{1}{4}(r_0 + \gamma m \text{Re } z)^2 + (\gamma m \text{Im } z)^2$$

and

$$(\text{Im } \rho)^2 - (\text{Re } \rho)^2 = r_0 - \gamma m \text{Re } z \geq 0$$

When $z \in Z_2$ these inequalities clearly hold for all $r_0 \geq 0$. □

Let $\phi \in C^\infty(\mathbf{R})$, $\phi(\sigma) = 1$ for $|\sigma| \leq 1$, $\phi(\sigma) = 0$ for $|\sigma| \geq 2$, and set

$$\chi(x', \xi') = \phi(\delta_0 r_0(x', \xi'))$$

where $0 < 2\delta_0 \leq \min_{x' \in \Gamma} \frac{1}{\gamma m(x')}$. We will say that a function $a \in C^\infty(T^*\Gamma)$ belongs to $S_{\delta_1, \delta_2}^{\ell_1}(\mu_1) + S_{\delta_3, \delta_4}^{\ell_2}(\mu_2)$ if $\chi a \in S_{\delta_1, \delta_2}^{\ell_1}(\mu_1)$ and $(1 - \chi)a \in S_{\delta_3, \delta_4}^{\ell_2}(\mu_2)$.

Lemma 3.2 *We have $\rho, |\rho| \in S_{2,2}^1(|\rho|) + S_{0,1}^1(|\rho|)$, $\rho^{-1}, |\rho|^{-1} \in S_{2,2}^{-1}(|\rho|) + S_{0,1}^{-1}(|\rho|)$ uniformly in z .*

Proof. In view of Lemma 3.1, we have $|\rho| \leq C$ on $\text{supp } \chi$, $|\rho| \geq C$ on $\text{supp } (1 - \chi)$, $C > 0$. We have to show that the function ρ satisfies the estimates

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta \rho \right| \leq C_{\alpha, \beta} |\rho|^{1-2|\alpha|-2|\beta|} \quad \text{on } \text{supp } \chi, \quad (3.5)$$

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta \rho \right| \leq C_{\alpha, \beta} |\rho|^{1-|\beta|} \quad \text{on } \text{supp } (1 - \chi), \quad (3.6)$$

for all multi-indices α and β with constants $C_{\alpha, \beta} > 0$ independent of z , and similarly for the function $|\rho|$. We will proceed by induction in $K = |\alpha| + |\beta|$. Differentiating the above equation we get

$$E_{\alpha, \beta} := -\partial_{x'}^\alpha \partial_{\xi'}^\beta (r_0(x', \xi') - \gamma m(x')z) = \partial_{x'}^\alpha \partial_{\xi'}^\beta (\rho^2) = 2\rho \partial_{x'}^\alpha \partial_{\xi'}^\beta \rho + F_{\alpha, \beta}$$

where $F_{\alpha, \beta}$ is a linear combination of functions of the form $\partial_{x'}^{\alpha_1} \partial_{\xi'}^{\beta_1} \rho \partial_{x'}^{\alpha_2} \partial_{\xi'}^{\beta_2} \rho$ with multi-indices satisfying $|\alpha_1| + |\alpha_2| = |\alpha|$, $|\beta_1| + |\beta_2| = |\beta|$, $|\alpha_1| + |\beta_1| \leq K - 1$, $|\alpha_2| + |\beta_2| \leq K - 1$. Hence, assuming (3.5) and (3.6) fulfilled for $|\alpha| + |\beta| \leq K - 1$ leads to the conclusion that $F_{\alpha, \beta} = O(|\rho|^{2-2|\alpha|-2|\beta|})$ on $\text{supp } \chi$ and $F_{\alpha, \beta} = O(|\rho|^{2-|\beta|})$ on $\text{supp } (1 - \chi)$. On the other hand, we have $E_{\alpha, \beta} = O(1)$ on $\text{supp } \chi$ and $E_{\alpha, \beta} = O(|\rho|^{2-|\beta|})$ on $\text{supp } (1 - \chi)$ uniformly in z . From this and the above identity we conclude that (3.5) and (3.6) hold for $|\alpha| + |\beta| = K$, as desired. The proof concerning the function $|\rho|$ is similar, using the identity

$$\begin{aligned} \tilde{E}_{\alpha, \beta} &:= \partial_{x'}^\alpha \partial_{\xi'}^\beta \left((r_0(x', \xi') - \gamma m(x') \text{Re } z)^2 + (\gamma m(x') \text{Im } z)^2 \right) \\ &= \partial_{x'}^\alpha \partial_{\xi'}^\beta (|\rho|^4) = 4|\rho|^3 \partial_{x'}^\alpha \partial_{\xi'}^\beta |\rho| + \tilde{F}_{\alpha, \beta} \end{aligned}$$

where $\tilde{F}_{\alpha, \beta}$ is a linear combination of functions of the form $\partial_{x'}^{\alpha_1} \partial_{\xi'}^{\beta_1} |\rho| \partial_{x'}^{\alpha_2} \partial_{\xi'}^{\beta_2} |\rho| \partial_{x'}^{\alpha_3} \partial_{\xi'}^{\beta_3} |\rho| \partial_{x'}^{\alpha_4} \partial_{\xi'}^{\beta_4} |\rho|$ with multi-indices satisfying $|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4| = |\alpha|$, $|\beta_1| + |\beta_2| + |\beta_3| + |\beta_4| = |\beta|$, $|\alpha_j| + |\beta_j| \leq |\alpha| + |\beta| - 1$, $j = 1, 2, 3, 4$. Clearly, on $\text{supp } \chi$ we have $\tilde{E}_{\alpha, \beta} = O(|\rho|^2)$ for $|\alpha| + |\beta| = 1$, $\tilde{E}_{\alpha, \beta} = O(1)$ for $|\alpha| + |\beta| \geq 2$, while on $\text{supp } (1 - \chi)$ we have $\tilde{E}_{\alpha, \beta} = O(|\rho|^{4-|\beta|})$. Therefore, the estimates (3.5) and (3.6) for the function $|\rho|$ can be proved by induction in $|\alpha| + |\beta|$ as above. The function ρ^{-1} (resp. $|\rho|^{-1}$) can be treated similarly using the identity

$$0 = \partial_{x'}^\alpha \partial_{\xi'}^\beta (\rho \rho^{-1}) = \rho \partial_{x'}^\alpha \partial_{\xi'}^\beta (\rho^{-1}) + \mathcal{F}_{\alpha, \beta}$$

for $|\alpha| + |\beta| \geq 1$, where $\mathcal{F}_{\alpha, \beta}$ is a linear combination of functions of the form $\partial_{x'}^{\alpha_1} \partial_{\xi'}^{\beta_1} \rho \partial_{x'}^{\alpha_2} \partial_{\xi'}^{\beta_2} (\rho^{-1})$ with multi-indices satisfying $|\alpha_1| + |\alpha_2| = |\alpha|$, $|\beta_1| + |\beta_2| = |\beta|$, $|\alpha_j| + |\beta_j| \leq |\alpha| + |\beta| - 1$, $j = 1, 2$. \square

Denote $\mathcal{D}_\nu = -ih\partial_\nu$, $Z_{1, \varepsilon} := \{z \in Z_1 : |\text{Im } z| \geq h^{\frac{1}{2}-\varepsilon}\}$, where $0 \leq \varepsilon \ll 1$. We also equip the Sobolev space $H^1(\Gamma)$ with the semi-classical norm $\|f\|_{H^1(\Gamma)} = \sum_{|\alpha| \leq 1} h^{|\alpha|} \|\partial_{x'}^\alpha f\|_{L^2(\Gamma)}$.

Theorem 3.3 *Given any $0 < \epsilon \ll 1$ there is $0 < h_0(\epsilon) \ll 1$ so that for $z \in Z_{1, \epsilon}$ and $0 < h \leq h_0$ the solution u to (3.1) satisfies the estimate*

$$\|\gamma \mathcal{D}_\nu u - \text{Op}_h(\rho + hb)f\|_{H^1(\Gamma)} \leq \frac{Ch}{\sqrt{|\text{Im } z|}} \|f\|_{L^2(\Gamma)} \quad (3.7)$$

where $b \in S_{0,1}^0(\langle \xi' \rangle)$ does not depend on h , z and the function n . Moreover, (3.7) holds for all $z \in Z_2 \cup Z_3$ with $|\text{Im } z|$ replaced by 1.

Proof. To prove (3.7) we will construct a parametrix to the solution of (3.1) near the boundary Γ . In fact, it suffices to carry out this construction locally and then to glue up all pices by using a partition of the unity on Γ . Indeed, it is well-known that given an arbitrary point $x^0 \in \Gamma$, there exists a small neighbourhood $\mathcal{O}(x^0) \subset \overline{\Omega}$ of x^0 and local coordinates $(x_1, x') \in \mathcal{O}(x^0)$ such that $x^0 = (0, 0)$, $\Gamma \cap \mathcal{O}(x^0)$ is defined by $x_1 = 0$, x' being coordinates in $\Gamma \cap \mathcal{O}(x^0)$, $x_1 > 0$ in $\Omega \cap \mathcal{O}(x^0)$, and in these coordinates the operator

$$\mathcal{P}(z, h) = -\frac{h^2}{c(x)} \nabla c(x) \nabla - z \frac{n(x)}{c(x)}$$

can be written in the form

$$\mathcal{P}(z, h) = \mathcal{D}_{x_1}^2 + r(x, \mathcal{D}_{x'}) - zm(x) + hq(x, \mathcal{D}_x) + h^2 \tilde{q}(x),$$

where we have put $\mathcal{D}_{x_1} = -ih\partial_{x_1}$, $\mathcal{D}_{x'} = -ih\partial_{x'}$, $r(x, \xi') = \langle R(x)\xi', \xi' \rangle$, $R = (R_{ij})$ being a symmetric $(d-1) \times (d-1)$ matrix-valued function with smooth real-valued entries, $q(x, \xi) = \langle q(x), \xi \rangle$, $q(x)$ and $\tilde{q}(x)$ being smooth functions. Moreover, we have $r(0, x', \xi') = r_0(x', \xi')$, the principal symbol of $-\Delta_\Gamma$ written in the coordinates (x', ξ') . Let $\psi(x') \in C_0^\infty(\Gamma \cap \mathcal{O}(x^0))$, $\psi = 1$ in a neighbourhood of x^0 . We will construct a parametrix, \tilde{u}_ψ , of (3.1), $\tilde{u}_\psi|_{x_1=0} = \psi f$, in the form

$$\tilde{u}_\psi(x) = (2\pi h)^{-d+1} \int \int e^{\frac{i}{h}\varphi(x, y', \xi', z)} \Phi_\delta(x, \xi', z) a(x, \xi', z, h) f(y') dy' d\xi',$$

where $\Phi_\delta = \phi\left(\frac{x_1}{\delta}\right) \phi\left(\frac{x_1}{\delta\rho_1}\right)$, $\rho_1 = 1$ if $z \in Z_2 \cup Z_3$, $\rho_1 = |\rho|^3$ if $z \in Z_1$, ϕ being as above, $\delta > 0$ is a small constant independent of x, ξ', h, z to be fixed later on. The phase φ is a complex-valued function such that $\varphi|_{x_1=0} = -\langle x' - y', \xi' \rangle$, and the amplitude a satisfies $a|_{x_1=0} = \psi(x')$. More generally, given any integer $N \gg 1$ we will be searching φ and a in the form

$$\varphi = -\langle x' - y', \xi' \rangle + \sum_{k=1}^{N-1} x_1^k \varphi_k(x', \xi', z),$$

$$a = \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} x_1^k h^j a_{k,j}(x', \xi', z)$$

so that φ satisfies the eikonal equation mod $O(x_1^N)$:

$$(\partial_{x_1} \varphi)^2 + r(x, \nabla_{x'} \varphi) - m(x)z = x_1^N \Psi_N(x, \xi', z) \quad (3.8)$$

and a satisfies the equation

$$e^{-\frac{i}{h}\varphi} \mathcal{P}(z, h) e^{\frac{i}{h}\varphi} a = x_1^N A_N(x, \xi', z, h) + h^N B_N(x, \xi', z, h) \quad (3.9)$$

where Ψ_N , A_N and B_N are smooth functions. In Section 4 we will prove the following

Proposition 3.4 *Let $z \in Z_{1,0} \cup Z_2 \cup Z_3$. Then, for a suitable choice of the constant δ , the equations (3.8) and (3.9) have smooth solutions φ and a of the form above, $\varphi = -\langle x' - y', \xi' \rangle + \tilde{\varphi}$, with $\varphi_1 = \rho$, $a_{0,0} = \psi$, $a_{0,j} = 0$ for $j \geq 1$, $a_{1,j} \in S_{2,2}^{-1-4j}(|\rho|) + S_{0,1}^{-j}(|\rho|)$, $j \geq 0$,*

$$a_{1,0} = -\frac{i}{2} q(0, x', 1, \xi'/\rho) \psi - \frac{1}{2\rho} \langle R(0, x') \xi', \nabla_{x'} \psi(x') \rangle.$$

Moreover, for all integers $k \geq 0$, we have

$$x_1^{-1} \tilde{\varphi} \in S_{2,2}^1(|\rho|) + S_{0,1}^1(|\rho|), \quad \partial_{x_1}^k \tilde{\varphi} \in S_{2,2}^{4-3k}(|\rho|) + S_{0,1}^1(|\rho|),$$

$$\partial_{x_1}^k a \in S_{2,2}^{2-3k}(|\rho|) + S_{0,1}^0(|\rho|),$$

$$\partial_{x_1}^k A_N \in S_{2,2}^{2-3N-3k}(|\rho|) + S_{0,1}^2(|\rho|), \quad \partial_{x_1}^k B_N \in S_{2,2}^{3-4N-3k}(|\rho|) + S_{0,1}^{1-N}(|\rho|),$$

with respect to the variables x', ξ' uniformly in z, h and $0 \leq x_1 \leq 2\delta \min\{1, \rho_1\}$. Finally, for $0 < x_1 \leq 2\delta \min\{1, \rho_1\}$ we have $\text{Im } \varphi \geq x_1 \text{Im } \rho / 2$.

Define the sets $\mathcal{M}_j \subset Z \times T^*\Gamma$, $j = 1, 2$, as follows: $\mathcal{M}_1 := Z_{1,0} \times \text{supp } \chi$, $\mathcal{M}_2 := Z_1 \times \text{supp } (1 - \chi) \cup Z_2 \times T^*\Gamma \cup Z_3 \times T^*\Gamma$. It follows from Lemma 3.1 that if $(z, x', \xi') \in \mathcal{M}_1$, then $C\sqrt{|\text{Im } z|} \leq |\rho| \leq \tilde{C}$ and $\text{Im } \rho \geq \frac{|\text{Im } z|}{2|\rho|}$, while for $(z, x', \xi') \in \mathcal{M}_2$ we have $C_1 \langle \xi' \rangle \leq |\rho| \leq C_2 \langle \xi' \rangle$ and $\text{Im } \rho \geq C \langle \xi' \rangle$.

Clearly, we have $\mathcal{D}_{x_1} \tilde{u}_\psi|_{x_1=0} = T_\psi(z, h)f = \text{Op}_h(\tau_\psi)f$, where

$$\tau_\psi = a \frac{\partial \varphi}{\partial x_1} \Big|_{x_1=0} - ih \frac{\partial a}{\partial x_1} \Big|_{x_1=0} = \psi \rho - ih \sum_{j=0}^{N-1} h^j a_{1,j}.$$

Lemma 3.5 *If $z \in Z_{1,0}$ we have the estimate*

$$\|T_\psi(z, h)f - \text{Op}_h(\psi \rho + hb_\psi)f\|_{H^1(\Gamma)} \leq \frac{Ch}{\sqrt{|\text{Im } z|}} \|f\|_{L^2(\Gamma)} \quad (3.10)$$

where

$$b_\psi = -\frac{i}{2}(1 - \chi)\psi q(0, x', 1, \xi' / \sqrt{r_0(x', \xi')}) - \frac{1}{2}(1 - \chi)\langle R(0, x')\xi' / \sqrt{r_0(x', \xi')}, \nabla_{x'} \psi(x') \rangle.$$

Moreover, (3.10) holds for all $z \in Z_2 \cup Z_3$ with $|\text{Im } z|$ replaced by 1.

Proof. If $z \in Z_{1,0}$, it follows from the above proposition that $\sum_{j=0}^{N-1} h^j \chi a_{1,j} \in S_{2,2}^{-1}(\sqrt{|\text{Im } z|})$, and hence by Proposition 2.1,

$$\begin{aligned} \left\| \text{Op}_h \left(\sum_{j=0}^{N-1} h^j \chi a_{1,j} \right) f \right\|_{H^1(\Gamma)} &\leq C \left\| \text{Op}_h \left(\sum_{j=0}^{N-1} h^j \chi a_{1,j} \right) f \right\|_{L^2(\Gamma)} \\ &\leq \frac{C}{\sqrt{|\text{Im } z|}} \left(1 + \frac{\sqrt{h}}{|\text{Im } z|} \right)^d \|f\|_{L^2(\Gamma)} \leq \frac{\tilde{C}}{\sqrt{|\text{Im } z|}} \|f\|_{L^2(\Gamma)} \end{aligned}$$

as long as $|\text{Im } z| \geq \sqrt{h}$. Clearly, the above bound holds for all $z \in Z_2 \cup Z_3$ with $|\text{Im } z|$ replaced by 1. On the other hand, it is easy to see that

$$(a_{1,0} - ib_\psi)(1 - \chi) + \sum_{j=1}^{N-1} h^j (1 - \chi) a_{1,j} \in S_{0,1}^{-1}(\langle \xi' \rangle) = \mathcal{S}_0^{-1}$$

uniformly in z and h . Hence the h -psdo with this symbol is bounded from L^2 to H^1 . \square

Proposition 3.6 *Let u_ψ satisfy $(P(h) - z)u_\psi = 0$ in Ω , $u_\psi|_\Gamma = \psi f$. Then, if $z \in Z_{1,0}$,*

$$\|\gamma \mathcal{D}_\nu u_\psi - T_\psi(z, h)f\|_{H^1(\Gamma)} \leq C_N h^{-s_d} \left(\frac{\sqrt{h}}{|\operatorname{Im} z|} \right)^{2N} \|f\|_{L^2(\Gamma)} \quad (3.11)$$

with constants $C_N, s_d > 0$ independent of f, h and z , s_d independent of N . If $z \in Z_2 \cup Z_3$, then (3.11) holds with $|\operatorname{Im} z|$ replaced by 1.

Proof. Given an integer $s \geq 0$, $H^s(\Omega)$ will denote the Sobolev space equipped with the semi-classical norm

$$\|g\|_{H^s(\Omega)} = \sum_{|\alpha| \leq s} \|\mathcal{D}_x^\alpha g\|_{L^2(\Omega)}.$$

Denote also by G_D the Dirichlet self-adjoint realization of the operator $-n^{-1}\nabla c \nabla$ on the Hilbert space $L^2(\Omega, n(x)dx)$. Then the function

$$w_\psi := u_\psi - \tilde{u}_\psi + \left(h^2 G_D - z \right)^{-1} \frac{c}{n} \mathcal{P}(z, h) \tilde{u}_\psi$$

satisfies the equation $(h^2 G_D - z) w_\psi = 0$ in Ω , $w_\psi|_\Gamma = 0$. Since z/h^2 does not belong to the spectrum of G_D , this implies that w_ψ is identically zero. Thus we get

$$\begin{aligned} \|\gamma \mathcal{D}_\nu u_\psi - \gamma \mathcal{D}_\nu \tilde{u}_\psi\|_{H^1(\Gamma)} &\leq \left\| \gamma \mathcal{D}_\nu \left(h^2 G_D - z \right)^{-1} \frac{c}{n} \mathcal{P}(z, h) \tilde{u}_\psi \right\|_{H^1(\Gamma)} \\ &\leq C h^{-1/2} \left\| \left(h^2 G_D - z \right)^{-1} \frac{c}{n} \mathcal{P}(z, h) \tilde{u}_\psi \right\|_{H^4(\Omega)} \end{aligned} \quad (3.12)$$

where we have used the semi-classical version of the trace theorem. On the other hand, it is well known that the resolvent of the operator G_D satisfies the bound

$$\left\| \left(h^2 G_D - z \right)^{-1} \right\|_{H^{2k}(\Omega) \rightarrow H^{2k}(\Omega)} \leq \frac{C_k}{|\operatorname{Im} z|} \quad (3.13)$$

for every integer $k \geq 0$. Indeed, for $k = 0$ (3.13) is trivial, while for $k \geq 1$ it follows from the coercive estimate

$$\|v\|_{H^{2k}(\Omega)} \leq \tilde{C}_k \left\| h^2 G_D v \right\|_{H^{2k-2}(\Omega)} + \tilde{C}_k \|v\|_{H^{2k-2}(\Omega)}, \quad \forall v \in D(G_D) \cap H^{2k-2}(\Omega).$$

Thus, (3.11) follows from (3.12), (3.13) and the following

Proposition 3.7 *If $z \in Z_{1,0}$, given any integer $s \geq 0$ there are $\ell_s, N_s > 0$ so that for $N \geq N_s$ we have the estimate*

$$\|\mathcal{P}(z, h) \tilde{u}_\psi\|_{H^s(\Omega)} \leq C_N h^{-\ell_s} \left(\frac{\sqrt{h}}{|\operatorname{Im} z|} \right)^{2N} \|f\|_{L^2(\Gamma)}. \quad (3.14)$$

If $z \in Z_2 \cup Z_3$, then (3.14) holds with $|\operatorname{Im} z|$ replaced by 1.

Proof. In view of (3.9) we can write

$$\mathcal{P}(z, h)\tilde{u}_\psi = (2\pi h)^{-d+1} \int \int e^{-\frac{i}{h}\langle x' - y', \xi' \rangle} K(x, \xi', z, h) f(y') dy' d\xi',$$

where

$$K = e^{\frac{i}{h}\langle x', \xi' \rangle} [\mathcal{P}(z, h), \Phi_\delta] e^{-\frac{i}{h}\langle x', \xi' \rangle} e^{\frac{i}{h}\tilde{\varphi}} a + e^{\frac{i}{h}\tilde{\varphi}} \Phi_\delta \left(x_1^N A_N + h^N B_N \right) =: K_1 + K_2.$$

Lemma 3.8 *If $z \in Z_{1,0}$, for any multi-index α there are $\ell_\alpha, N_\alpha > 0$ so that for $N \geq N_\alpha$ we have*

$$|\partial_x^\alpha K| \leq C_{\alpha, N} h^{-\ell_\alpha} \left(\frac{\sqrt{h}}{|\operatorname{Im} z|} \right)^{2N}. \quad (3.15)$$

If $z \in Z_2 \cup Z_3$, then (3.15) holds with $|\operatorname{Im} z|$ replaced by 1.

Proof. An easy computation leads to the identity

$$\begin{aligned} [\mathcal{P}(z, h), \Phi_\delta] &= -2ih \frac{\partial \Phi_\delta}{\partial x_1} \mathcal{D}_{x_1} - 2ih \langle R(x) \nabla_{x'} \Phi_\delta, \mathcal{D}_{x'} \rangle \\ &\quad - h^2 \frac{\partial^2 \Phi_\delta}{\partial x_1^2} - h^2 \sum_{ij} R_{ij}(x) \frac{\partial^2 \Phi_\delta}{\partial x'_i \partial x'_j} - ih^2 q(x, \nabla_x \Phi_\delta). \end{aligned}$$

Hence

$$\begin{aligned} e^{\frac{i}{h}\langle x', \xi' \rangle} [\mathcal{P}(z, h), \Phi_\delta] e^{-\frac{i}{h}\langle x', \xi' \rangle} &= -2ih \frac{\partial \Phi_\delta}{\partial x_1} \mathcal{D}_{x_1} - 2ih \langle R(x) \nabla_{x'} \Phi_\delta, \mathcal{D}_{x'} \rangle \\ &\quad - h^2 \frac{\partial^2 \Phi_\delta}{\partial x_1^2} - h^2 \sum_{ij} R_{ij}(x) \frac{\partial^2 \Phi_\delta}{\partial x'_i \partial x'_j} - ih^2 q(x, \nabla_x \Phi_\delta) - 2ih \langle R(x) \nabla_{x'} \Phi_\delta, \xi' \rangle. \end{aligned}$$

Observe now that if $|\alpha| \geq 1$, then the function $\partial_x^\alpha \Phi_\delta$ is supported in the region $\Theta := \delta \min\{1, \rho_1\} \leq x_1 \leq 2\delta \min\{1, \rho_1\}$. To prove (3.15) we will consider two cases.

Case 1. $(z, x', \xi') \in \mathcal{M}_1$. Then $\rho_1 = |\rho|^3$ and $C'|\rho|^3 \leq \min\{1, \rho_1\} \leq |\rho|^3$. It is easy to see that in this case we have $\partial_x^\alpha \Phi_\delta = O(|\operatorname{Im} z|^{-\ell_\alpha}) = O(h^{-\ell_\alpha/2})$ as long as $|\operatorname{Im} z| \geq \sqrt{h}$. In view of Proposition 3.4, on Θ we also have

$$\begin{aligned} \left| e^{\frac{i}{h}\tilde{\varphi}} a \right| &\leq \tilde{C} \exp(-\operatorname{Im} \tilde{\varphi}/h) \leq \tilde{C} \exp\left(-\frac{x_1 |\operatorname{Im} z|}{2h|\rho|}\right) \\ &\leq \tilde{C} \exp\left(-C|\rho|^2 |\operatorname{Im} z|/h\right) \leq \tilde{C} \exp\left(-C|\operatorname{Im} z|^2/h\right) \end{aligned}$$

and more generally

$$\left| \partial_x^\alpha \left(e^{\frac{i}{h}\tilde{\varphi}} a \right) \right| \leq \tilde{C}_\alpha h^{-\ell_\alpha} \exp\left(-C|\operatorname{Im} z|^2/h\right).$$

Thus we get

$$|\partial_x^\alpha K_1| \leq \tilde{C}_\alpha h^{-\ell_\alpha} \exp\left(-C|\operatorname{Im} z|^2/h\right) \quad (3.16)$$

with probably new constants. Furthermore, on $\operatorname{supp} \Phi_\delta$ we have

$$\left| x_1^N e^{\frac{i}{h}\tilde{\varphi}} \right| \leq \tilde{C} x_1^N \exp\left(-\frac{x_1 |\operatorname{Im} z|}{2h|\rho|}\right) \leq C_N \left(\frac{h|\rho|}{|\operatorname{Im} z|} \right)^N.$$

On the other hand, by Proposition 3.4 we have $A_N = O_N(|\rho|^{-3N})$, $B_N = O_N(|\rho|^{-4N})$. Hence

$$\left| x_1^N e^{\frac{i}{h}\tilde{\varphi}} A_N \right| + \left| h^N e^{\frac{i}{h}\tilde{\varphi}} B_N \right| \leq C_N \left(\frac{h}{|\rho|^2 |\operatorname{Im} z|} \right)^N + C_N \left(\frac{h}{|\rho|^4} \right)^N \leq C_N \left(\frac{\sqrt{h}}{|\operatorname{Im} z|} \right)^{2N}.$$

Moreover, it is easy to see that differentiating these two functions makes appear additional factors $O(|\operatorname{Im} z|^{-\ell_1} h^{-\ell_2}) = O(h^{-\ell_1/2 - \ell_2})$ with ℓ_1 and ℓ_2 depending only on the order of differentiation. Thus we get

$$|\partial_x^\alpha K_2| \leq C_N h^{-\ell_\alpha} \left(\frac{\sqrt{h}}{|\operatorname{Im} z|} \right)^{2N}. \quad (3.17)$$

Clearly, in this case (3.15) follows from (3.16) and (3.17).

Case 2. $(z, x', \xi') \in \mathcal{M}_2$. Then $\min\{1, \rho_1\} \geq \text{Const} > 0$. As above, it is easy to see that in this case we have

$$|\partial_x^\alpha K_1| \leq \tilde{C}_\alpha \left(h^{-1} \langle \xi' \rangle \right)^{\ell_\alpha} \exp \left(-\frac{C \langle \xi' \rangle}{h} \right) \quad (3.18)$$

and

$$|\partial_x^\alpha K_2| \leq C_N \left(h^{-1} \langle \xi' \rangle \right)^{\ell_\alpha} \left(\frac{h}{\langle \xi' \rangle} \right)^N. \quad (3.19)$$

If $N \geq \ell_\alpha$ we deduce from these bounds that $\partial_x^\alpha K_1, \partial_x^\alpha K_2 = O_N(h^{N-\ell'_\alpha})$, which again implies (3.15). \square

It follows from Proposition 2.1 and Lemma 3.8 that, for $z \in Z_{1,0}$,

$$\|\partial_x^\alpha \mathcal{P}(z, h) \tilde{u}_\psi(x_1, \cdot)\|_{L^2(\Gamma)} \leq C_N h^{-\tilde{\ell}_\alpha} \left(\frac{\sqrt{h}}{|\operatorname{Im} z|} \right)^{2N} \|f\|_{L^2(\Gamma)}$$

for $0 \leq x_1 \leq 2\delta$. Hence

$$\begin{aligned} \|\partial_x^\alpha \mathcal{P}(z, h) \tilde{u}_\psi\|_{L^2(\Omega)} &\leq \left(\int_0^{2\delta} \|\partial_x^\alpha \mathcal{P}(z, h) \tilde{u}_\psi(x_1, \cdot)\|_{L^2(\Gamma)}^2 dx_1 \right)^{1/2} \\ &\leq \sqrt{2\delta} C_N h^{-\tilde{\ell}_\alpha} \left(\frac{\sqrt{h}}{|\operatorname{Im} z|} \right)^{2N} \|f\|_{L^2(\Gamma)} \end{aligned}$$

which clearly implies (3.14) in this case. If $z \in Z_2 \cup Z_3$, the above estimates clearly hold with $|\operatorname{Im} z|$ replaced by 1. \square

Let $\{\psi_j\}_{j=1}^J$ be a partition of the identity on Γ . Then we have $u = \sum_{j=1}^J u_{\psi_j}$ and $T(z, h) = \sum_{j=1}^J T_{\psi_j}(z, h)$ is an h -psdo on Γ with a principal symbol ρ . Observe that if $z \in Z_{1,\epsilon}$, there are $N_0 = N_0(\epsilon) \gg 1$ and $h_0 = h_0(\epsilon) \ll 1$ such that for $N \geq N_0$ and $0 < h \leq h_0$ we have

$$C_N h^{-s_d} \left(\frac{\sqrt{h}}{|\operatorname{Im} z|} \right)^{2N} \leq C_N h^{2\epsilon N - s_d} \leq h.$$

This bound clearly holds for all $z \in Z_2 \cup Z_3$ (with $|\operatorname{Im} z|$ replaced by 1). Therefore, (3.7) follows from (3.10) and (3.11) with $b = \sum_{j=1}^J b_{\psi_j}$. \square

In what follows, given any $s \in \mathbf{R}$ we denote $\|f\|_{H^s(\Gamma)} := \|\operatorname{Op}_h(\langle \xi' \rangle^s) f\|_{L^2(\Gamma)}$.

Lemma 3.9 *Let $z \in Z_2$. Then we have*

$$\left\| \frac{dT}{dz}(z, h) f - \operatorname{Op}_h\left(\frac{d\rho}{dz}(z)\right) f \right\|_{L^2(\Gamma)} \leq Ch \|f\|_{H^{-1}(\Gamma)} \quad (3.20)$$

with a constant $C > 0$ independent of z , h and f . Moreover,

$$\left| \operatorname{Re} \langle cT(-1, h) f, f \rangle_{L^2(\Gamma)} \right| \leq C_N h^{N-s_d} \|f\|_{L^2(\Gamma)}^2. \quad (3.21)$$

Proof. It follows from Lemma 4.3 below that

$$\sum_{j=0}^{N-1} h^j \frac{da_{1,j}}{dz} \in S_{0,1}^{-1}(\langle \xi' \rangle) = \mathcal{S}_0^{-1}.$$

Hence the h -psdo with this symbol is bounded from H^{-1} to L^2 uniformly in h and $z \in Z_2$, which implies (3.20). To prove (3.21) observe that by Green's formula we have the identity

$$\operatorname{Im} \langle c\partial_\nu \tilde{u}|_\Gamma, f \rangle_{L^2(\Gamma)} = -\operatorname{Im} \langle \nabla c \nabla \tilde{u}, \tilde{u} \rangle_{L^2(\Omega)} = -\operatorname{Im} \langle (\nabla c \nabla - h^{-2}n) \tilde{u}, \tilde{u} \rangle_{L^2(\Omega)}$$

where $\tilde{u} = \sum_{j=1}^J \tilde{u}_{\psi_j}$. Hence

$$\left| \operatorname{Re} \langle cT(-1, h) f, f \rangle_{L^2(\Gamma)} \right| \leq h^{-1} \|\mathcal{P}(-1, h) \tilde{u}\|_{L^2(\Omega)} \|\tilde{u}\|_{L^2(\Omega)}. \quad (3.22)$$

By Proposition 2.1 it is easy to see that $\|\tilde{u}\|_{L^2(\Omega)} \leq Ch^{-s'_d} \|f\|_{L^2(\Gamma)}$, which together with (3.14) and (3.22) imply (3.21). \square

4 Proof of Proposition 3.4

We will first solve equation (3.8). We can expand the functions $R(x)$ and $m(x)$ as follows

$$\begin{aligned} R(x) &= \sum_{k=0}^{N-1} x_1^k R_k(x') + x_1^N \mathcal{R}_N(x), \\ m(x) &= \sum_{k=0}^{N-1} x_1^k m_k(x') + x_1^N M_N(x), \end{aligned}$$

where R_k , \mathcal{R}_N , m_k , M_N are smooth functions. Thus, if $\varphi = \sum_{k=0}^{N-1} x_1^k \varphi_k(x')$, we have

$$\begin{aligned} \mathcal{E} &:= (\partial_{x_1} \varphi)^2 + \langle R(x) \nabla_{x'} \varphi, \nabla_{x'} \varphi \rangle - zm(x) \\ &= \sum_{k=0}^{N-2} \sum_{j=0}^{N-2} (k+1)(j+1) x_1^{k+j} \varphi_{k+1} \varphi_{j+1} \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} x_1^{k+j} \langle R \nabla_{x'} \varphi_k, \nabla_{x'} \varphi_j \rangle - z \sum_{k=0}^{N-1} x_1^k m_k - z x_1^N M_N \\
& = \sum_{k+j \leq N-1} (k+1)(j+1) x_1^{k+j} \varphi_{k+1} \varphi_{j+1} \\
& + \sum_{k+j \leq N-1} x_1^{k+j} \langle R \nabla_{x'} \varphi_k, \nabla_{x'} \varphi_j \rangle - z \sum_{k=0}^{N-1} x_1^k m_k + x_1^N \Psi_N^{(1)}
\end{aligned}$$

where

$$\begin{aligned}
\Psi_N^{(1)} & = \sum_{k, j \leq N-2, k+j \geq N} (k+1)(j+1) x_1^{k+j-N} \varphi_{k+1} \varphi_{j+1} \\
& + \sum_{k, j \leq N-1, k+j \geq N} x_1^{k+j-N} \langle R \nabla_{x'} \varphi_k, \nabla_{x'} \varphi_j \rangle - z M_N.
\end{aligned}$$

We also have

$$\sum_{k+j \leq N-1} x_1^{k+j} \langle R \nabla_{x'} \varphi_k, \nabla_{x'} \varphi_j \rangle = \sum_{k+j+\ell \leq N-1} x_1^{k+j+\ell} \langle R_\ell \nabla_{x'} \varphi_k, \nabla_{x'} \varphi_j \rangle + x_1^N \Psi_N^{(2)}$$

where

$$\begin{aligned}
\Psi_N^{(2)} & = \sum_{k+j \leq N-1} x_1^{k+j} \langle \mathcal{R}_N \nabla_{x'} \varphi_k, \nabla_{x'} \varphi_j \rangle \\
& + \sum_{\ell \leq N-1, k+j \leq N-1, k+j+\ell \geq N} x_1^{k+j+\ell-N} \langle R_\ell \nabla_{x'} \varphi_k, \nabla_{x'} \varphi_j \rangle.
\end{aligned}$$

Thus we have $\mathcal{E} = x_1^N \Psi_N$ with $\Psi_N = \Psi_N^{(1)} + \Psi_N^{(2)}$, provided the coefficients φ_k satisfy the relationships

$$\sum_{k+j=K} (k+1)(j+1) \varphi_{k+1} \varphi_{j+1} + \sum_{k+j+\ell=K} \langle R_\ell \nabla_{x'} \varphi_k, \nabla_{x'} \varphi_j \rangle - z m_K = 0 \quad (4.1)$$

for every integer $0 \leq K \leq N-2$. Clearly, if we take $\varphi_0 = -\langle x' - y', \xi' \rangle$, then $\varphi_1 = \rho$ is a solution of (4.1) with $K = 0$. Now, given φ_j , $0 \leq j \leq K-1$, $K \geq 2$, we can determine φ_K in a unique way by (4.1).

Lemma 4.1 *We have $\varphi_k \in S_{2,2}^{4-3k}(|\rho|) + S_{0,1}^1(|\rho|)$, $1 \leq k \leq N-1$, $\partial_{x_1}^k \Psi_N \in S_{2,2}^{2-3N-3k}(|\rho|) + S_{0,1}^2(|\rho|)$, $k \geq 0$, uniformly in z and $0 \leq x_1 \leq 2\delta \min\{1, |\rho|^3\}$. Moreover, if $\delta > 0$ is small enough, independent of ρ , we have*

$$\text{Im } \varphi \geq x_1 \text{Im } \rho / 2 \quad \text{for } 0 \leq x_1 \leq 2\delta \min\{1, |\rho|^3\}. \quad (4.2)$$

Proof. In view of Lemma 3.2 we have $z m_K \rho^{-1} \in S_{2,2}^{-1}(|\rho|) + S_{0,1}^{-1}(|\rho|)$ uniformly in z . We will now proceed by induction. Suppose that $\varphi_k \in S_{2,2}^{4-3k}(|\rho|) + S_{0,1}^1(|\rho|)$, $1 \leq k \leq K$. This implies $\nabla_{x'} \varphi_k \in S_{2,2}^{2-3k}(|\rho|) + S_{0,1}^1(|\rho|)$, $1 \leq k \leq K$, which yields

$$\langle R_\ell \nabla_{x'} \varphi_k, \nabla_{x'} \varphi_j \rangle \in S_{2,2}^{4-3K}(|\rho|) + S_{0,1}^2(|\rho|), \quad k+j+\ell \leq K, \quad k, j \geq 1. \quad (4.3)$$

Furthermore, since $\nabla_{x'}\varphi_0 = \xi'$, we have

$$\langle R_\ell \nabla_{x'} \varphi_k, \nabla_{x'} \varphi_0 \rangle, \langle R_\ell \nabla_{x'} \varphi_0, \nabla_{x'} \varphi_j \rangle \in S_{2,2}^{2-3K}(|\rho|) + S_{0,1}^2(|\rho|), \quad 1 \leq k, j \leq K, \quad (4.4)$$

$$\langle R_\ell \nabla_{x'} \varphi_0, \nabla_{x'} \varphi_0 \rangle \in S_{2,2}^0(|\rho|) + S_{0,1}^2(|\rho|). \quad (4.5)$$

We also have

$$\varphi_{k+1} \varphi_{j+1} \in S_{2,2}^{2-3K}(|\rho|) + S_{0,1}^2(|\rho|), \quad k+j=K, \quad k, j \geq 1. \quad (4.6)$$

Thus by equation (4.1) and (4.3)-(4.6) we conclude that $2\rho\varphi_{K+1} - zm_K \in S_{2,2}^{2-3K}(|\rho|) + S_{0,1}^2(|\rho|)$, and hence $\varphi_{K+1} \in S_{2,2}^{1-3K}(|\rho|) + S_{0,1}^1(|\rho|)$ as desired. The property concerning the function Ψ_N follows easily from the following observation: if $k \geq 0$ and $a \in S_{2,2}^{\ell_1}(|\rho|) + S_{0,1}^{\ell_2}(|\rho|)$, then for $0 \leq x_1 \leq 2\delta \min\{1, |\rho|^3\}$ we have $x_1^k a \in S_{2,2}^{\ell_1+3k}(|\rho|) + S_{0,1}^{\ell_2}(|\rho|)$.

To bound $\text{Im } \varphi$ from below we will show that for every multi-index α we have the estimate

$$|\text{Im } \partial_{x'}^\alpha \varphi_k| \leq \frac{C_{k,\alpha} \text{Im } \rho}{\min\{1, |\rho|^{3k-3+2|\alpha|}\}}, \quad k \geq 1. \quad (4.7)$$

For $(z, x', \xi') \in \mathcal{M}_2$ we have

$$|\text{Im } \partial_{x'}^\alpha \varphi_k| \leq |\partial_{x'}^\alpha \varphi_k| \leq \tilde{C}_{k,\alpha} |\rho| \leq C_{k,\alpha} \text{Im } \rho$$

which implies (4.7) in this case. Let now $(z, x', \xi') \in \mathcal{M}_1$. Observe first that

$$|\text{Im}(z\rho^{-1})| \leq |\text{Im } z| |\rho|^{-1} + |z| |\text{Im}(\rho^{-1})| \leq 2\text{Im } \rho + 2|\rho|^{-2} \text{Im } \rho \leq C|\rho|^{-2} \text{Im } \rho.$$

Differentiating equation (4.1) we obtain

$$\begin{aligned} & 2(K+1)\partial_{x'}^{\alpha_1} \varphi_{K+1} + \rho^{-1} \sum_{\Theta_{K,\alpha}} (k+1)(j+1) \partial_{x'}^{\alpha_1} \varphi_{k+1} \partial_{x'}^{\alpha_2} \varphi_{j+1} \\ & + \rho^{-1} \sum_{\tilde{\Theta}_{K,\alpha}} \langle \partial_{x'}^{\alpha_1} R_\ell \partial_{x'}^{\alpha_2} \nabla_{x'} \varphi_k, \partial_{x'}^{\alpha_3} \nabla_{x'} \varphi_j \rangle - z\rho^{-1} \partial_{x'}^\alpha m_K = 0 \end{aligned} \quad (4.8)$$

where $\Theta_{K,\alpha} := \{(k, j, \alpha_1, \alpha_2) : k+j=K, k, j \leq K-1, |\alpha_1|+|\alpha_2|=|\alpha|, |\alpha_1|, |\alpha_2| \leq |\alpha|-1\}$, $\tilde{\Theta}_{K,\alpha} := \{(\ell, k, j, \alpha_1, \alpha_2, \alpha_3) : \ell+k+j=K, |\alpha_1|+|\alpha_2|+|\alpha_3|=|\alpha|\}$. To prove (4.7) in this case we will proceed by induction in k and $|\alpha|$. Fix integers $K \geq 1$ and $A \geq 0$ and suppose that (4.7) holds for $1 \leq k \leq K$ and all α , and for $k=K+1$ and $|\alpha| \leq A-1$. We have to show that (4.7) holds for $k=K$ and $|\alpha|=A$. To this end we will use (4.8). Observe that on $\Theta_{K,\alpha}$ we have

$$\begin{aligned} & \left| \text{Im} \left(\rho^{-1} \partial_{x'}^{\alpha_1} \varphi_{k+1} \partial_{x'}^{\alpha_2} \varphi_{j+1} \right) \right| \leq \left| \text{Im} \left(\rho^{-1} \right) \right| |\partial_{x'}^{\alpha_1} \varphi_{k+1}| |\partial_{x'}^{\alpha_2} \varphi_{j+1}| \\ & + |\rho|^{-1} |\text{Im } \partial_{x'}^{\alpha_1} \varphi_{k+1}| |\partial_{x'}^{\alpha_2} \varphi_{j+1}| + |\rho|^{-1} |\partial_{x'}^{\alpha_1} \varphi_{k+1}| |\text{Im } \partial_{x'}^{\alpha_2} \varphi_{j+1}| \leq C_{K,\alpha} |\rho|^{-3K-2A} \text{Im } \rho \end{aligned} \quad (4.9)$$

where we have used our hypothesis and the fact that in this case $\varphi_k \in S_{2,2}^{4-3k}(|\rho|)$. Similarly, on $\tilde{\Theta}_{K,\alpha}$, $k, j \geq 1$, we have

$$\left| \text{Im} \left(\rho^{-1} \partial_{x'}^{\alpha_2} \partial_{x'} \varphi_k \partial_{x'}^{\alpha_3} \partial_{x'} \varphi_j \right) \right| \leq C_{K,\alpha} |\rho|^{2-3K-2A} \text{Im } \rho. \quad (4.10)$$

If one of k or j is 0, since $\nabla_{x'}\varphi_0 = \xi'$ is bounded on $\text{supp } \chi$, the left-hand side of (4.10) is $O(|\rho|^{-3K-2A}\text{Im } \rho)$, while for $j = k = 0$ it is $O(\text{Im } (\rho^{-1})) = O(|\rho|^{-2}\text{Im } \rho)$. Thus, by (4.8) we conclude that

$$|\text{Im } \partial_{x'}^\alpha \varphi_{K+1}| \leq C_{K,\alpha} |\rho|^{-3K-2A} \text{Im } \rho$$

which is the desired bound.

Using (4.7) with $\alpha = 0$ we obtain, for $0 < x_1 \leq 2\delta \min\{1, |\rho|^3\}$,

$$\begin{aligned} \text{Im } \varphi &\geq x_1 \text{Im } \rho - x_1 \sum_{k=2}^{N-1} x_1^{k-1} |\text{Im } \varphi_k| \geq x_1 \text{Im } \rho - x_1 \text{Im } \rho \sum_{k=2}^{N-1} C_k x_1^{k-1} (\min\{1, |\rho|^3\})^{-k+1} \\ &\geq x_1 \text{Im } \rho (1 - O(\delta)) \geq x_1 \text{Im } \rho / 2 \end{aligned}$$

provided $\delta > 0$ is taken small enough, independent of ρ . \square

To solve equation (3.9) observe first that

$$\begin{aligned} \mathcal{Q} &:= e^{-\frac{i}{h}\varphi} \mathcal{P}(z, h) e^{\frac{i}{h}\varphi} = \mathcal{P}(z, h) \\ &+ 2 \frac{\partial \varphi}{\partial x_1} \mathcal{D}_{x_1} + 2 \langle R(x) \nabla_{x'} \varphi, \mathcal{D}_{x'} \rangle + h q(x, \nabla_x \varphi) + \left(\frac{\partial \varphi}{\partial x_1} \right)^2 + r(x, \nabla_{x'} \varphi) \\ &= \mathcal{D}_{x_1}^2 + r(x, \mathcal{D}_{x'}) + h q(x, \mathcal{D}_x) + h^2 \tilde{q}(x) \\ &+ 2 \frac{\partial \varphi}{\partial x_1} \mathcal{D}_{x_1} + 2 \langle R(x) \nabla_{x'} \varphi, \mathcal{D}_{x'} \rangle + h q(x, \nabla_x \varphi) + x_1^N \Psi_N \end{aligned}$$

where we have used that the phase function satisfies equation (3.8). Write

$$q(x, \xi) = \sum_{k=0}^{N-1} x_1^k q_k(x', \xi) + x_1^N Q_N(x, \xi),$$

$$q_k(x', \xi) = q_k^\sharp(x') \xi_1 + q_k^\flat(x', \xi'),$$

$$\tilde{q}(x) = \sum_{k=0}^{N-1} x_1^k \tilde{q}_k(x') + x_1^N \tilde{Q}_N(x).$$

We will be searching a solution to (3.9) in the form $a = \sum_{j=0}^{N-1} h^j a_j(x, z)$, $a_0|_{x_1=0} = \psi$, $a_j|_{x_1=0} = 0$, $j \geq 1$. Thus, if the functions a_j satisfy the transport equations

$$\begin{aligned} &-2i \frac{\partial \varphi}{\partial x_1} \frac{\partial a_j}{\partial x_1} - 2i \langle R(x) \nabla_{x'} \varphi, \nabla_{x'} a_j \rangle + q(x, \nabla_x \varphi) a_j \\ &= \left(\partial_{x_1}^2 + r(x, \partial_{x'}) + i q(x, \partial_x) - \tilde{q}(x) \right) a_{j-1} + x_1^N A_N^{(j)}, \quad 0 \leq j \leq N-1, \end{aligned} \quad (4.11)$$

$a_{-1} = 0$, then

$$\mathcal{Q}a = x_1^N \sum_{j=0}^N h^j A_N^{(j-1)} - h^N \left(\partial_{x_1}^2 + r(x, \partial_{x'}) + i q(x, \partial_x) - \tilde{q}(x) \right) a_{N-1} = x_1^N A_N + h^N B_N \quad (4.12)$$

where we have put $A_N^{(-1)} = \Psi_N$. We will be looking for solutions of (4.11) in the form $a_j = \sum_{k=0}^{N-1} x_1^k a_{k,j}$, $a_{0,0} = \psi$, $a_{0,j} = 0$, $j \geq 1$. We have

$$\begin{aligned}
& \frac{\partial \varphi}{\partial x_1} \frac{\partial a_j}{\partial x_1} = \sum_{\nu+k \leq N-1} x_1^{\nu+k} (\nu+1)(k+1) \varphi_{\nu+1} a_{k+1,j} \\
& + x_1^N \sum_{\nu+k \geq N, \nu, k \leq N-1} x_1^{\nu+k-N} (\nu+1)(k+1) \varphi_{\nu+1} a_{k+1,j}, \\
& \langle R(x) \nabla_{x'} \varphi, \nabla_{x'} a_j \rangle = \sum_{\ell+\nu+k \leq N-1} x_1^{\ell+\nu+k} \langle R_\ell(x') \nabla_{x'} \varphi_\nu, \nabla_{x'} a_{k,j} \rangle \\
& + x_1^N \sum_{\ell+\nu+k \geq N} x_1^{\ell+\nu+k-N} \langle R_\ell(x') \nabla_{x'} \varphi_\nu, \nabla_{x'} a_{k,j} \rangle + x_1^N \langle \mathcal{R}_N(x) \nabla_{x'} \varphi, \nabla_{x'} a_j \rangle, \\
& q(x, \nabla_x \varphi) a_j = \sum_{\ell=0}^{N-1} x_1^\ell q_\ell(x', \nabla_x \varphi) a_j + x_1^N Q_N(x, \nabla_x \varphi) a_j \\
& = \sum_{\ell=0}^{N-1} x_1^\ell \left(q_\ell^\sharp(x') \partial_{x_1} \varphi + q_\ell^\flat(x', \nabla_{x'} \varphi) \right) a_j + x_1^N Q_N(x, \nabla_x \varphi) a_j \\
& = \sum_{\ell+\nu+k \leq N-1} x_1^{\ell+\nu+k} \left((\nu+1) q_\ell^\sharp(x') \varphi_{\nu+1} + q_\ell^\flat(x', \nabla_{x'} \varphi_\nu) \right) a_{k,j} \\
& + x_1^N \sum_{\ell+\nu+k \geq N} x_1^{\ell+\nu+k-N} \left((\nu+1) q_\ell^\sharp(x') \varphi_{\nu+1} + q_\ell^\flat(x', \nabla_{x'} \varphi_\nu) \right) a_{k,j} + x_1^N Q_N(x, \nabla_x \varphi) a_j, \\
& \left(\partial_{x_1}^2 + r(x, \partial_{x'}) + i q(x, \partial_x) - \tilde{q}(x) \right) a_{j-1} \\
& = \sum_{k=0}^{N-2} x_1^k (k+2)(k+1) a_{k+2,j-1} + \sum_{\ell+k \leq N-1} x_1^{\ell+k} \langle R_\ell(x') \nabla_{x'}, \nabla_{x'} a_{k,j-1} \rangle \\
& + \sum_{\ell+k \leq N-1} x_1^{k+\ell} \left((k+1) i q_\ell^\sharp a_{k+1,j-1} + i q_\ell^\flat(x', \nabla_{x'} a_{k,j-1}) - \tilde{q}_\ell a_{k,j-1} \right) \\
& + x_1^N \sum_{\ell+k \geq N} x_1^{\ell+k-N} \langle R_\ell(x') \nabla_{x'}, \nabla_{x'} a_{k,j-1} \rangle \\
& + x_1^N \sum_{\ell+k \geq N} x_1^{k+\ell-N} \left((k+1) i q_\ell^\sharp a_{k+1,j-1} + i q_\ell^\flat(x', \nabla_{x'} a_{k,j-1}) - \tilde{q}_\ell a_{k,j-1} \right) \\
& + x_1^N \langle \mathcal{R}_N(x') \nabla_{x'}, \nabla_{x'} a_{j-1} \rangle + x_1^N \left(i Q_N(x, \nabla_x a_{j-1}) - \tilde{Q}_N a_{j-1} \right).
\end{aligned}$$

Thus we obtain that the coefficients $a_{k,j}$ must satisfy the equations

$$\begin{aligned}
& -2i \sum_{\nu+k=K} (\nu+1)(k+1) \varphi_{\nu+1} a_{k+1,j} - 2i \sum_{\ell+\nu+k=K} \langle R_\ell(x') \nabla_{x'} \varphi_\nu, \nabla_{x'} a_{k,j} \rangle \\
& + \sum_{\ell+\nu+k=K} \left((\nu+1) q_\ell^\sharp(x') \varphi_{\nu+1} + q_\ell^\flat(x', \nabla_{x'} \varphi_\nu) \right) a_{k,j} \\
& = (K+2)(K+1) a_{K+2,j-1} + \sum_{\ell+k=K} \langle R_\ell(x') \nabla_{x'}, \nabla_{x'} a_{k,j-1} \rangle
\end{aligned}$$

$$+ \sum_{\ell+k=K} \left((k+1)iq_\ell^\sharp a_{k+1,j-1} + iq_\ell^\flat(x', \nabla_{x'} a_{k,j-1}) - \tilde{q}_\ell a_{k,j-1} \right) \quad (4.13)$$

for every integer $0 \leq K \leq N-2$. Clearly, there exist unique solutions $a_{k,j}$ of (4.13) such that $a_{0,0} = \psi$, $a_{0,j} = 0$, $j \geq 1$, and $a_{k,-1} = 0$, $k \geq 0$.

Lemma 4.2 *We have $a_{k,j} \in S_{2,2}^{2-3k-4j}(|\rho|) + S_{0,1}^{-j}(|\rho|)$, $k \geq 1$, $j \geq 0$, $\partial_{x_1}^k A_N^{(j)} \in S_{2,2}^{-3N-4j-3k}(|\rho|) + S_{0,1}^1(|\rho|)$, $j \geq 0$, $\partial_{x_1}^k A_N \in S_{2,2}^{2-3N-3k}(|\rho|) + S_{0,1}^2(|\rho|)$, $\partial_{x_1}^k B_N \in S_{2,2}^{3-4N-3k}(|\rho|) + S_{0,1}^{1-N}(|\rho|)$, $k \geq 0$.*

Proof. Observe first that equation (4.13) with $K = 0$, $j = 0$ yields the formula

$$a_{1,0} = -\frac{i}{2}q(0, x', 1, \xi'/\rho)\psi - \frac{1}{2\rho}\langle R(0, x')\xi', \nabla_{x'}\psi(x') \rangle.$$

Hence $a_{1,0} \in S_{2,2}^{-1}(|\rho|) + S_{0,1}^0(|\rho|)$. To prove the assertion concerning the functions $a_{k,j}$ we will proceed by induction in k and j . Fix $K \geq 1$, $J \geq 1$ and suppose that our assertion is true for all $0 \leq j \leq J-1$, $k \geq 1$, and for $j = J$ and $1 \leq k \leq K$. We have to show that it is true for $j = J$ and $k = K+1$. Our hypothesis together with Lemma 4.1 imply

$$\varphi_{\nu+1} a_{k+1,J} \in S_{2,2}^{-3K-4J}(|\rho|) + S_{0,1}^{1-J}(|\rho|), \quad \nu + k = K, \quad 0 \leq k \leq K-1,$$

$$\langle R_\ell(x') \nabla_{x'} \varphi_\nu, \nabla_{x'} a_{k,J} \rangle \in S_{2,2}^{2-3K-4J}(|\rho|) + S_{0,1}^{1-J}(|\rho|), \quad \ell + \nu + k = K, \quad \nu \geq 1,$$

$$\langle R_\ell(x') \nabla_{x'} \varphi_0, \nabla_{x'} a_{k,J} \rangle \in S_{2,2}^{-3K-4J}(|\rho|) + S_{0,1}^{1-J}(|\rho|), \quad \ell + k = K,$$

$$q_\ell^\sharp(x') \varphi_{\nu+1} a_{k,J} \in S_{2,2}^{3-3K-4J}(|\rho|) + S_{0,1}^{1-J}(|\rho|), \quad \ell + \nu + k = K,$$

$$q_\ell^\flat(x', \nabla_{x'} \varphi_\nu) a_{k,J} \in S_{2,2}^{4-3K-4J}(|\rho|) + S_{0,1}^{1-J}(|\rho|), \quad \ell + \nu + k = K, \quad \nu \geq 1,$$

$$q_\ell^\flat(x', \nabla_{x'} \varphi_0) a_{k,J} \in S_{2,2}^{2-3K-4J}(|\rho|) + S_{0,1}^{1-J}(|\rho|), \quad \ell + k = K.$$

One can also easily see that the right-hand side of equation (4.13) belongs to $S_{2,2}^{-3K-4J}(|\rho|) + S_{0,1}^{1-J}(|\rho|)$. Thus, by (4.13) we conclude that $\rho a_{K+1,J} \in S_{2,2}^{-3K-4J}(|\rho|) + S_{0,1}^{1-J}(|\rho|)$, which implies $a_{K+1,J} \in S_{2,2}^{-1-3K-4J}(|\rho|) + S_{0,1}^{-J}(|\rho|)$, as desired. The properties concerning the functions $A_N^{(j)}$, A_N , B_N follow easily from the following observation: if $k \geq 0$, $j \geq 0$, and $a \in S_{2,2}^{\ell_1}(|\rho|) + S_{0,1}^{\ell_2}(|\rho|)$, then for $0 \leq x_1 \leq 2\delta \min\{1, |\rho|^3\}$ we have $h^j x_1^k a \in S_{2,2}^{\ell_1+3k+4j}(|\rho|) + S_{0,1}^{\ell_2}(|\rho|)$, where we have used that $h \leq |\operatorname{Im} z|^2 \leq C|\rho|^4$. \square

Lemma 4.3 *Let $z \in Z_2$. Then $\frac{d\varphi_k}{dz}, \frac{da_{k,j}}{dz} \in S_{0,1}^{-1}(\langle \xi' \rangle)$, $k \geq 1$, $j \geq 0$.*

Proof. Recall that in this case we have $C_1 \langle \xi' \rangle \leq |\rho| \leq C_2 \langle \xi' \rangle$. We also have $\frac{d\varphi_0}{dz} = 0$ and $2\rho \frac{d\rho}{dz} = -m_0(x')$. Hence $\frac{d\rho}{dz} \in S_{0,1}^{-1}(\langle \xi' \rangle)$. Differentiating equation (4.1) once with respect to the variable z it is easy to see that $\rho \frac{d\varphi_{K+1}}{dz} \in S_{0,1}^0(\langle \xi' \rangle)$, provided $\frac{d\varphi_k}{dz} \in S_{0,1}^{-1}(\langle \xi' \rangle)$ for $1 \leq k \leq K$, which implies $\frac{d\varphi_{K+1}}{dz} \in S_{0,1}^{-1}(\langle \xi' \rangle)$. Thus we obtain the desired properties of the functions $\frac{d\varphi_k}{dz}$ by induction in k . Similalry, we have $\frac{da_{0,j}}{dz} = 0$, $j \geq 0$, and $\frac{da_{1,0}}{dz} \in S_{0,1}^{-1}(\langle \xi' \rangle)$. Differentiating equation (4.13) once with respect to the variable z it is easy to see that $\rho \frac{da_{K+1,J}}{dz} \in S_{0,1}^0(\langle \xi' \rangle)$, provided $\frac{da_{k,j}}{dz} \in S_{0,1}^{-1}(\langle \xi' \rangle)$ for $0 \leq j \leq J-1$, $k \geq 1$, and $j = J$, $1 \leq k \leq K$. Therefore, the desired result follows by induction in j and k . \square

5 Eigenvalue-free regions

In this section we will study the problem

$$\begin{cases} (P_1(h) - z) u_1 = 0 & \text{in } \Omega, \\ (P_2(h) - z) u_2 = 0 & \text{in } \Omega, \\ u_1 = u_2, \quad c_1 \partial_\nu u_1 = c_2 \partial_\nu u_2 & \text{on } \Gamma, \end{cases} \quad (5.1)$$

where $z \in Z$, $0 < h \ll 1$, $P_j(h)$, $j = 1, 2$, is defined by replacing in the definition of the operator $P(h)$ from Section 3 the pair (c, n) by (c_j, n_j) . Similarly, we define the functions ρ_j by replacing in the definition of ρ the function m by $m_j = \frac{n_j}{c_j}$. We will also use the function χ introduced at the beginning of Section 3. Note that we can make the support of χ as large as we want by taking the parameter δ_0 small enough. It follows from Theorem 3.3 that if $z \in Z_{1,\epsilon}$, the function $f := u_1|_\Gamma = u_2|_\Gamma$ satisfies the estimate

$$\|\text{Op}_h(c_1 \rho_1 - c_2 \rho_2) f\|_{L^2(\Gamma)} \leq \frac{Ch}{\sqrt{|\text{Im } z|}} \|f\|_{L^2(\Gamma)} \quad (5.2)$$

while in the case $c_1|_\Gamma \equiv c_2|_\Gamma$ we have the better estimate

$$\|\text{Op}_h(\rho_1 - \rho_2) f\|_{H^1(\Gamma)} \leq \frac{Ch}{\sqrt{|\text{Im } z|}} \|f\|_{L^2(\Gamma)}. \quad (5.3)$$

Moreover, (5.2) and (5.3) hold for all $z \in Z_2 \cup Z_3$ with $|\text{Im } z|$ replaced by 1. We would like to invert the operators in the left-hand sides of (5.2) and (5.3). Note that it follows from Lemmas 3.1 and 3.2 that the function ρ_j satisfies the bounds, for $(z, x', \xi') \in \mathcal{M}_1$,

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta \rho_j \right| \leq C_{\alpha,\beta} |\text{Im } z|^{\frac{1}{2} - |\alpha| - |\beta|}, \quad |\alpha| + |\beta| \geq 1, \quad (5.4)$$

$|\rho_j| \leq \text{Const}$, while for $(z, x', \xi') \in \mathcal{M}_2$ we have

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta \rho_j \right| \leq C_{\alpha,\beta} \langle \xi' \rangle^{1 - |\beta|}. \quad (5.5)$$

In particular, these estimates imply that $\rho_j \in \mathcal{S}_{\frac{1}{2}-\epsilon}^1$ if $z \in Z_{1,\epsilon}$ and $\rho_j \in \mathcal{S}_0^1$ if $z \in Z_2 \cup Z_3$. Observe now that

$$c_1 \rho_1 - c_2 \rho_2 = \frac{\tilde{c}(x')(c_0(x') r_0(x', \xi') - z)}{c_1 \rho_1 + c_2 \rho_2} \quad (5.6)$$

where \tilde{c} and c_0 are the restrictions on Γ of the functions

$$c_1 n_1 - c_2 n_2 \quad \text{and} \quad \frac{c_1^2 - c_2^2}{c_1 n_1 - c_2 n_2}$$

respectively. It follows easily from (5.4)-(5.6) that

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta (c_1 \rho_1 - c_2 \rho_2) \right| \leq C_{\alpha,\beta} |\text{Im } z|^{\frac{1}{2} - |\alpha| - |\beta|}, \quad |\alpha| + |\beta| \geq 1, \quad (5.7)$$

for $(z, x', \xi') \in \mathcal{M}_1$, and

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta (c_1 \rho_1 - c_2 \rho_2) \right| \leq C_{\alpha,\beta} \langle \xi' \rangle^{k - |\beta|} \quad (5.8)$$

for $(z, x', \xi') \in \mathcal{M}_2$ and all multi-indices α and β , where $k = -1$ if $c_0 \equiv 0$, $k = 1$ if $c_0(x') \neq 0$, $\forall x' \in \Gamma$. In particular, these estimates imply that $c_1\rho_1 - c_2\rho_2 \in \mathcal{S}_{\frac{1}{2}-\epsilon}^k$ if $z \in Z_{1,\epsilon}$ and $c_1\rho_1 - c_2\rho_2 \in \mathcal{S}_0^k$ if $z \in Z_2 \cup Z_3$. We will now consider two cases.

Case 1. $c_0 \equiv 0$. Then $k = -1$. In this case we have $|\rho_1 - \rho_2| \geq C\langle \xi' \rangle^{-1}$, $C > 0$, so $(\rho_1 - \rho_2)^{-1} \in \mathcal{S}_{\frac{1}{2}-\epsilon}^1$ if $z \in Z_{1,\epsilon}$ and $(\rho_1 - \rho_2)^{-1} \in \mathcal{S}_0^1$ if $z \in Z_2 \cup Z_3$. Hence

$$\left\| \text{Op}_h \left((\rho_1 - \rho_2)^{-1} \right) g \right\|_{L^2(\Gamma)} \leq C \|g\|_{H^1(\Gamma)}, \quad \forall g \in H^1(\Gamma), \quad (5.9)$$

for $z \in Z_{1,\epsilon} \cup Z_2 \cup Z_3$. By (5.3) and (5.9), for $z \in Z_{1,\epsilon}$,

$$\left\| \text{Op}_h \left((\rho_1 - \rho_2)^{-1} \right) \text{Op}_h(\rho_1 - \rho_2) f \right\|_{L^2(\Gamma)} \leq \frac{Ch}{\sqrt{|\text{Im } z|}} \|f\|_{L^2(\Gamma)}. \quad (5.10)$$

For $z \in Z_2 \cup Z_3$, (5.10) holds with $|\text{Im } z|$ replaced by 1. On the other hand, by Proposition 2.2 we have

$$\left\| \text{Op}_h \left((\rho_1 - \rho_2)^{-1} \right) \text{Op}_h(\rho_1 - \rho_2) - Id \right\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq Ch^{2\epsilon}. \quad (5.11)$$

Combining (5.10) and (5.11) we conclude $\|f\|_{L^2} \leq O(h^{2\epsilon})\|f\|_{L^2}$ for $z \in Z_{1,\epsilon} \cup Z_2 \cup Z_3$, which implies $f \equiv 0$ provided h is taken small enough.

Case 2. $c_0(x') \neq 0$, $\forall x' \in \Gamma$. Then $k = 1$. Observe first that the condition (1.7) implies $c_0 > 0$. It is easy to see that if $z \in Z_2$, then we have $|c_1\rho_1 - c_2\rho_2| \geq C\langle \xi' \rangle$, $C > 0$, so $(c_1\rho_1 - c_2\rho_2)^{-1} \in \mathcal{S}_0^{-1}$. Hence

$$\left\| \text{Op}_h \left((c_1\rho_1 - c_2\rho_2)^{-1} \right) g \right\|_{L^2(\Gamma)} \leq C \|g\|_{L^2(\Gamma)}, \quad \forall g \in L^2(\Gamma). \quad (5.12)$$

By (5.2) and (5.12), if $z \in Z_2$,

$$\left\| \text{Op}_h \left((c_1\rho_1 - c_2\rho_2)^{-1} \right) \text{Op}_h(c_1\rho_1 - c_2\rho_2) f \right\|_{L^2(\Gamma)} \leq Ch \|f\|_{L^2(\Gamma)}. \quad (5.13)$$

On the other hand, by Proposition 2.2 we have

$$\left\| \text{Op}_h \left((c_1\rho_1 - c_2\rho_2)^{-1} \right) \text{Op}_h(c_1\rho_1 - c_2\rho_2) - Id \right\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq Ch^{2\epsilon}. \quad (5.14)$$

In the same way as above one can derive from (5.13) and (5.14) that $f \equiv 0$, provided h is taken small enough.

Under the conditions (1.2) and (1.4) only, we have $|c_0 r_0| \geq C|\xi'|^2$, $C > 0$. Given any $0 < \delta' \ll 1$ and any multi-indices α and β , by induction in $|\alpha| + |\beta|$ one can easily prove that the following estimates hold true:

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta \left((c_0 r_0 - z)^{-1} \right) \right| \leq C_{\alpha,\beta} |\text{Im } z|^{-1-|\alpha|-|\beta|} \quad (5.15)$$

for $|c_0 r_0 - \text{Re } z| \leq \delta'$, $\text{Im } z \neq 0$, and

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta \left((c_0 r_0 - z)^{-1} \right) \right| \leq C_{\alpha,\beta} \langle \xi' \rangle^{-2-|\beta|} \quad (5.16)$$

for $|c_0 r_0 - \text{Re } z| \geq \delta'$. By (5.4), (5.5), (5.6), (5.15) and (5.16),

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta \left(c_1\rho_1 - c_2\rho_2 \right)^{-1} \right| \leq C_{\alpha,\beta} |\text{Im } z|^{-1-|\alpha|-|\beta|} \quad (5.17)$$

for $(z, x', \xi') \in \mathcal{M}_1$, $|c_0 r_0 - \operatorname{Re} z| \leq \delta'$,

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta \left((c_1 \rho_1 - c_2 \rho_2)^{-1} \right) \right| \leq C_{\alpha, \beta} |\operatorname{Im} z|^{-\frac{1}{2} - |\alpha| - |\beta|}, \quad |\alpha| + |\beta| \geq 1, \quad (5.18)$$

for $(z, x', \xi') \in \mathcal{M}_1$, $|c_0 r_0 - \operatorname{Re} z| \geq \delta'$, and

$$\left| \partial_{x'}^\alpha \partial_{\xi'}^\beta \left((c_1 \rho_1 - c_2 \rho_2)^{-1} \right) \right| \leq C_{\alpha, \beta} \langle \xi' \rangle^{-1 - |\beta|} \quad (5.19)$$

for $(z, x', \xi') \in \mathcal{M}_2$. In particular, these estimates imply $|\operatorname{Im} z| (c_1 \rho_1 - c_2 \rho_2)^{-1} \in \mathcal{S}_{\frac{1}{2} - \epsilon}^{-1}$ for $z \in Z_{1, \epsilon}$ and $(c_1 \rho_1 - c_2 \rho_2)^{-1} \in \mathcal{S}_0^{-1}$ for $z \in Z_3$. Hence we have

$$\left\| \operatorname{Op}_h \left((c_1 \rho_1 - c_2 \rho_2)^{-1} \right) g \right\|_{L^2(\Gamma)} \leq \frac{C}{|\operatorname{Im} z|} \|g\|_{L^2(\Gamma)}, \quad \forall g \in L^2(\Gamma). \quad (5.20)$$

By (5.2) and (5.20),

$$\left\| \operatorname{Op}_h \left((c_1 \rho_1 - c_2 \rho_2)^{-1} \right) \operatorname{Op}_h (c_1 \rho_1 - c_2 \rho_2) f \right\|_{L^2(\Gamma)} \leq \frac{Ch}{|\operatorname{Im} z|^{3/2}} \|f\|_{L^2(\Gamma)}. \quad (5.21)$$

In view of (5.7), (5.8), (5.17)-(5.19), by Proposition 2.2 we have

$$\left\| \operatorname{Op}_h \left((c_1 \rho_1 - c_2 \rho_2)^{-1} \right) \operatorname{Op}_h (c_1 \rho_1 - c_2 \rho_2) - Id \right\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq \frac{Ch}{|\operatorname{Im} z|^{5/2}}. \quad (5.22)$$

Combining (5.21) and (5.22) leads to the inequality

$$\|f\|_{L^2(\Gamma)} \leq \frac{Ch}{|\operatorname{Im} z|^{3/2}} \|f\|_{L^2(\Gamma)} + \frac{Ch}{|\operatorname{Im} z|^{5/2}} \|f\|_{L^2(\Gamma)}. \quad (5.23)$$

Clearly, it follows from (5.23) that if h is taken small enough, for all $z \in Z_3$ and for $z \in Z_{1, \epsilon}$, $|\operatorname{Im} z| \geq C' h^{2/5}$, with a sufficiently large constant $C' > 0$, we have $\|f\|_{L^2} = 0$, as desired.

Consider now the case $z \in Z_{1, \epsilon}$ under the conditions (1.2), (1.4) and (1.5). It is easy to see that the condition (1.5) implies $\frac{c_j}{n_j}|_\Gamma \neq c_0$, $j = 1, 2$. Hence, if $\delta' > 0$ is taken small enough we can arrange that $|\rho_j| \geq \operatorname{Const} > 0$ on $|c_0 r_0 - 1| \leq \delta'$. Therefore, the functions $a^+ = (c_1 \rho_1 - c_2 \rho_2)^{-1}$ and $a^- = c_1 \rho_1 - c_2 \rho_2$ satisfy (2.4) with $\mu_0 = \frac{h}{|\operatorname{Im} z|^2}$, so Proposition 2.2 gives in this case (5.22) (and hence (5.23)) with $\frac{h}{|\operatorname{Im} z|^{5/2}}$ replaced by $\frac{h}{|\operatorname{Im} z|^2}$. Thus we obtain that $f \equiv 0$, provided $z \in Z_{1, \epsilon}$ and h taken small enough.

Under the conditions (1.2), (1.4) and (1.6), we have $\rho_1 \equiv \rho_2$. Hence (5.18) holds for all $(z, x', \xi') \in \mathcal{M}_1$, which again implies (2.4) with $\mu_0 = \frac{h}{|\operatorname{Im} z|^2}$, and the desired result follows as above.

It remains to consider the case $z \in Z_2$ under the condition (1.8). It suffices to consider the case $|\operatorname{Im} z| \leq \gamma_0$ with some constant $0 < \gamma_0 \ll 1$, since the case $\gamma_0 \leq |\operatorname{Im} z| \leq 1$ is easy and can be treated as the case $z \in Z_3$ above. By Proposition 3.6 we have

$$\|c_1 T_1(z, h) f - c_2 T_2(z, h) f\|_{L^2(\Gamma)} \leq C_N h^{N-s_d} \|f\|_{L^2(\Gamma)} \quad (5.24)$$

where T_j is defined by replacing in the definition of the operator $T(z, h)$ from Section 3 the functions c, n by c_j, n_j . Recall that in this case we have $|\rho_j| \geq C \langle \xi' \rangle$, which implies $c_1 T_1 - c_2 T_2 \in \operatorname{OPS}_0^1$. Since $|c_1 \rho_1 - c_2 \rho_2| \geq C \langle \xi' \rangle$ on $\operatorname{supp} (1 - \chi)$, we have

$$\|\operatorname{Op}_h (1 - \chi) g\|_{L^2(\Gamma)} \leq C \|(c_1 T_1 - c_2 T_2) g\|_{L^2(\Gamma)} + O_N(h^N) \|g\|_{L^2(\Gamma)} \quad (5.25)$$

for every $g \in L^2$ and $N \geq 1$. By (5.24) and (5.25),

$$\|\text{Op}_h(1 - \chi)f\|_{L^2(\Gamma)} \leq C_N h^{N-s_d} \|f\|_{L^2(\Gamma)}. \quad (5.26)$$

We will show that

$$|\text{Im } z| \|\text{Op}_h(\chi)f\|_{L^2(\Gamma)}^2 \leq C_N h^{N-s_d} \|f\|_{L^2(\Gamma)}^2. \quad (5.27)$$

To this end recall that $z = -1 + i\text{Im } z$. Clearly, there exists $0 < t \leq 1$ so that we can write

$$\begin{aligned} c_1 T_1(z, h) - c_2 T_2(z, h) &= c_1 T_1(-1, h)f - c_2 T_2(-1, h) \\ &\quad + i\text{Im } z \left(c_1 \frac{dT_1}{dz}(z_t, h) - c_2 \frac{dT_2}{dz}(z_t, h) \right) \end{aligned} \quad (5.28)$$

where $z_t = -1 + it\text{Im } z \in Z_2$. By Lemma 3.9 we have

$$\left| \text{Re } \langle (c_1 T_1(-1, h)f - c_2 T_2(-1, h))f, f \rangle_{L^2(\Gamma)} \right| \leq C_N h^{N-s_d} \|f\|_{L^2(\Gamma)}^2. \quad (5.29)$$

By (5.24), (5.28) and (5.29),

$$\begin{aligned} |\text{Im } z| &\left| \text{Im } \left\langle \left(c_1 \frac{dT_1}{dz}(z_t, h) - c_2 \frac{dT_2}{dz}(z_t, h) \right) f, f \right\rangle_{L^2(\Gamma)} \right| \\ &\leq \left| \text{Re } \langle (c_1 T_1(-1, h)f - c_2 T_2(-1, h))f, f \rangle_{L^2(\Gamma)} \right| \\ &\quad + \left| \text{Re } \langle (c_1 T_1(z, h)f - c_2 T_2(z, h))f, f \rangle_{L^2(\Gamma)} \right| \leq C_N h^{N-s_d} \|f\|_{L^2(\Gamma)}^2. \end{aligned} \quad (5.30)$$

It follows from (5.30) that to prove (5.27) it suffices to show that

$$\|\text{Op}_h(\chi)f\|_{L^2(\Gamma)}^2 \leq C \left| \text{Im } \left\langle \left(c_1 \frac{dT_1}{dz}(z, h) - c_2 \frac{dT_2}{dz}(z, h) \right) f, f \right\rangle_{L^2(\Gamma)} \right| \quad (5.31)$$

for every $z \in Z_2$ with a constant $C > 0$ independent of z and h . In view of Lemma 3.9 we have

$$\left\| \left(c_1 \frac{dT_1}{dz}(z, h) - c_2 \frac{dT_2}{dz}(z, h) \right) f - \text{Op}_h(\kappa(z))f \right\|_{L^2(\Gamma)} \leq Ch \|f\|_{H^{-1}(\Gamma)} \quad (5.32)$$

where

$$\begin{aligned} \kappa(z) &= c_1 \frac{d\rho_1(z)}{dz} - c_2 \frac{d\rho_2(z)}{dz} = -\frac{n_1}{2\rho_1(z)} + \frac{n_2}{2\rho_2(z)} \\ &= \frac{n_2^2 \rho_1^2 - n_1^2 \rho_2^2}{2\rho_1 \rho_2 (n_1 \rho_2 + n_2 \rho_1)} = \frac{c_1 c_2 (n_2^2 - n_1^2) r_0 - z n_1 n_2 (c_2 n_2 - c_1 n_1)}{2c_1 c_2 \rho_1 \rho_2 (n_1 \rho_2 + n_2 \rho_1)}. \end{aligned}$$

Clearly, we have $\kappa(z) \in \mathcal{S}_0^{-1}$ and

$$\frac{d\kappa(z)}{dz} = -\frac{n_1^2}{4\rho_1(z)^3} + \frac{n_2^2}{4\rho_2(z)^3} = O(\langle \xi' \rangle^{-3})$$

which implies

$$\kappa(z) = \kappa(-1) + O(|\text{Im } z| \langle \xi' \rangle^{-3}). \quad (5.33)$$

Since $\rho_j(-1) = i|\rho_j(-1)|$, we have

$$i\kappa(-1) = \frac{c_1c_2(n_1^2 - n_2^2)r_0 + n_1n_2(c_1n_1 - c_2n_2)}{2c_1c_2|\rho_1||\rho_2|(n_1|\rho_2| + n_2|\rho_1|)}.$$

On the other hand, it is easy to see that the condition (1.7) implies

$$(n_1(x) - n_2(x))(c_1(x)n_1(x) - c_2(x)n_2(x)) > 0, \quad \forall x \in \Gamma,$$

which in turn implies

$$\left| c_1c_2(n_1^2 - n_2^2)r_0 + n_1n_2(c_1n_1 - c_2n_2) \right| \geq C\langle \xi' \rangle^2$$

and hence

$$|\operatorname{Im} \kappa(-1)| = |\kappa(-1)| \geq C\langle \xi' \rangle^{-1}. \quad (5.34)$$

By (5.33) and (5.34),

$$|\operatorname{Im} \kappa(z)| \geq C\langle \xi' \rangle^{-1} \quad (5.35)$$

provided $|\operatorname{Im} z| \leq \gamma_0$ with some constant $0 < \gamma_0 \ll 1$. Clearly, we have

$$\operatorname{Im} \langle \operatorname{Op}_h(\kappa(z))f, f \rangle_{L^2(\Gamma)} = \langle \mathcal{A}f, f \rangle_{L^2(\Gamma)}$$

where $\mathcal{A} = (2i)^{-1}(\operatorname{Op}_h(\kappa(z)) - \operatorname{Op}_h(\kappa(z))^*)$ is an h -psdo belonging to $\operatorname{OPS}_0^{-1}$ with principal symbol $\operatorname{Im} \kappa(z)$. Since the function $\operatorname{Im} \kappa(z)$ is of constant sign, we can use Gårding's inequality together with (5.35) to obtain

$$\left| \langle \mathcal{A}f, f \rangle_{L^2(\Gamma)} \right| \geq C\|f\|_{H^{-1/2}(\Gamma)}^2, \quad C > 0. \quad (5.36)$$

Using that

$$\|\operatorname{Op}_h(\chi)f\|_{L^2(\Gamma)} \leq C\|f\|_{H^{-1}(\Gamma)} \leq C\|f\|_{H^{-1/2}(\Gamma)}$$

it is easy to see that (5.31) follows from (5.32) and (5.36), provided that h is taken small enough. By (5.26) and (5.27) we conclude

$$|\operatorname{Im} z| \|f\|_{L^2(\Gamma)}^2 \leq C_N h^{N-s_d} \|f\|_{L^2(\Gamma)}^2. \quad (5.37)$$

If $|\operatorname{Im} z| \geq 2C_N h^{N-s_d}$, we deduce from (5.37) that $\|f\|_{L^2} = 0$. Since $N \gg 1$ is arbitrary, this implies the desired result in this case. \square

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